1112 模組06-12班 微積分3 期考解答和評分標準

1. Consider the function

$$f(x,y)=\begin{cases}\dfrac{x^4y^2}{x^4+y^6} & \text{if } (x,y)\neq(0,0)\\ 0 & \text{if } (x,y)=(0,0)\end{cases}.$$

(a) (4%) Find $\lim_{(x,y)\to(0,0)} f(x,y)$. Is $f(x,y)$ continuous at $(0,0)$?

(b) (4%) Let $\mathbf{u}=\langle a,b\rangle$ be a unit vector (i.e. $a^2+b^2=1$). Use the definition of directional derivative to find $D_{\mathbf{u}}f(0,0)$.

(c) (2%) Write down the linearization $L(x,y)$ of $f(x,y)$ at $(0,0)$.

(d) (4%) Determine whether $f(x,y)$ is differentiable at $(0,0)$ by considering the limit

$$\lim_{(x,y)\to(0,0)}\frac{f(x,y)-L(x,y)}{\sqrt{x^2+y^2}}.$$

Solution:

Consider the function

$$f(x,y)=\begin{cases}\dfrac{x^4y^2}{x^4+y^6}, & (x,y)\neq(0,0)\\ 0, & (x,y)=(0,0)\end{cases}$$

(a) Is f continuous at $(0,0)$? **Sol.** We have that

$$0\le \underbrace{\frac{x^4}{x^4+y^6}}_{\le 1} y^2 \le y^2 \to 0, \quad \text{as } (x,y)\to(0,0)$$

by squeeze theorem, we have that

$$\lim_{(x,y)\to(0,0)} f(x,y)=0$$

thus f is continuous at origin.

Grading Policy.

- For squeeze:
  - Plug $\dfrac{x^4y^2}{x^4+y^6}$ into the limit and try to evaluate: [1%].
  - Trying to squeeze. [1%]
  - Correctly get the answer [2%], if try the right squeeze but with some minor error, [1%] will be taken away.
- Try polar coordinate: [3%] utmost.
  - Plug $\dfrac{x^4y^2}{x^4+y^6}$ into limit and try to evaluate [1%]. For switch to polar coordinate: [1%].
  - Try to argue that the function is bounded [1%]. Notice that in double variable case, the θ maybe depends on r, that is, $\theta=\theta(r)$[1]. Many student argue that $r^2\cos^4\theta\sin^2\theta\to 0, r^2\sin^6\theta\to 0$ as $r\to 0$, thus

    $$\lim_{r\to 0}\frac{r^2\cos^4\theta\sin^2\theta}{\cos^4\theta+r^2\sin^6\theta}=\frac{0}{\cos^4\theta+0}=\frac{1}{\cos^4\theta(0)}$$

    but notice that if $\theta(r)\to\dfrac{\pi}{2}\Rightarrow\cos\theta\to 0$ (not exactly 0 but approach 0!), then, the limit becomes a indefinite form, and one can not simply conclude that the limit exists unless he/she use the squeeze theorem.
  - If just write:

    $$\lim_{r\to 0}\frac{r^2\cos^4\theta\sin^2\theta}{\cos^4\theta+r^2\sin^6\theta}=0$$

    without any explanation, one utmost get [2%].
- Trying a particular path and conclude that function is continuous: utmost [1%]

(b) Let $\mathbf{u} = \langle a, b\rangle$ with $\mathbf{u}$ be unit vector, find $D_{\mathbf{u}}f(0,0)$ by definition.

Sol. By definition of directional derivative, we have that

$$\begin{aligned} D_{\mathbf{u}}f(0,0) &\overset{(i)}{=\!=} \lim_{h\to 0}\frac{f(0+ah,0+bh)-f(0,0)}{h} \\ &\overset{(ii)}{=\!=} \lim_{h\to 0}\frac{h^5a^4b^2}{h^4a^4+h^6b^6} \overset{(iii)}{=\!=} \lim_{h\to 0}\frac{ha^4b^2}{a^4+h^2b^6} \\ &\overset{(iv)}{=\!=} \begin{cases} \dfrac{\lim_{h\to 0} ha^4b^2}{a^4+\lim_{h\to 0} h^2b^6} = \dfrac{0}{a^4+0} = 0, & a\neq 0 \\ \lim\limits_{h\to 0}\dfrac{h\cdot 0\cdot b^2}{0+h^2b^6} = 0, & a = 0 \end{cases} \end{aligned}$$

Grading Policy.

- Each marked equality admits [1%].
- For those who use the identity: $D_{\mathbf{u}}f(0,0) = \nabla f_{(0,0)}\cdot\mathbf{u}$, [0%]. Since the conclusion can be made if f is differentiable, but up to problem (b), f is not verified whether it is differentiable.

(c) Write down the linearization $L(x,y)$ of $f(x,y)$ at $(0,0)$.

Sol. From (b), we have that

$$L(x,y) = f(0,0) + \underbrace{\frac{\partial f}{\partial x}(0,0)}_{=D_{\langle 1,0\rangle}f(0,0)=0}(x-0) + \underbrace{\frac{\partial f}{\partial y}(0,0)}_{=D_{\langle 0,1\rangle}f(0,0)=0}(y-0) = 0$$

Grading Policy.

- Write down the expression: $L(x,y) = f(0,0) + f_x(0,0)x + f_y(0,0)y$: [1%]. Use result in (b): [1%].
- If one does not explicitly say that the value of partial derivative is from part (b), or dose not calculate the value, then, [1%] will be taken away.
- If the value of partial derivative is not calculated, [1%] will be taken away.

(d) Determine whether $f(x,y)$ is differentiable at (0, 0) by considering the limit

$$\lim_{(x,y)\to(0,0)}\frac{f(x,y)-L(x,y)}{\sqrt{x^2+y^2}}$$

Sol. By part (c), $L(x,y) = 0$, we have that

$$\lim_{(x,y)\to(0,0)}\frac{f(x,y)-L(x,y)}{\sqrt{x^2+y^2}} = \lim_{(x,y)\to(0,0)}\frac{x^4y^2}{(x^4+y^6)\sqrt{x^2+y^2}}$$

notice that

$$0 \le \frac{x^4y^2}{(x^4+y^6)\sqrt{x^2+y^2}} = \underbrace{\frac{x^4}{x^4+y^6}}_{\le 1}\underbrace{\frac{|y|}{\sqrt{x^2+y^2}}}_{\le 1}|y| \le |y| \to 0, \quad \text{as } (x,y)\to 0$$

by squeeze theorem, we conclude that $\lim\limits_{(x,y)\to(0,0)}\dfrac{f(x,y)-L(x,y)}{\sqrt{x^2+y^2}} = 0$, thus f is differentiable.

Grading Policy. Most are likely to part (a)

- For squeeze:
  - Plug $\dfrac{x^4y^2}{(x^4+y^6)\sqrt{x^2+y^2}}$ into the limit and try to evaluate: [1%].
  - No matter what procedure is made, if the conclusion is that f is differentiable: [1%].
  - Correctly get the answer [2%], if try the right squeeze but with some minor error, [1%] will be taken away.
- Try polar coordinate: [3%] utmost.

- Plug $\dfrac{x^4y^2}{(x^4+y^6)\sqrt{x^2+y^2}}$ into limit, switch to polar coordinate

$$\frac{r\cos^4\theta\sin^2\theta}{\cos^4\theta+r^2\sin^6\theta}$$

- No matter what procedure is made, if the conclusion is that f is differentiable: [1%].
- Try to argue that the function is bounded [1%]. Notice again that in double variable case, the θ maybe depends on r, Many student argue that $r\cos^4\theta\sin^2\theta \to 0, r^2\sin^6\theta \to 0$ as $r \to 0$, thus

$$\lim_{r\to 0}\frac{r\cos^4\theta\sin^2\theta}{\cos^4\theta+r^2\sin^6\theta} = \frac{0}{\cos^4\theta+0} = \frac{1}{\cos^4\theta(0)}$$

  but notice that if $\theta(r) \to \dfrac{\pi}{2} \Rightarrow \cos\theta \to 0$ (not exactly 0 but approach 0!), then, the limit becomes a indefinite form, and one can not simply conclude that the limit exists unless he/she use the squeeze theorem.

2. A group of atmospheric scientists conducts an experiment in a valley. It is known that the surface V of the valley is given by the equation :

$$V : z^2x^4 + zy^6 = x^2 + y^4 + 4.$$

The scientists discover that the air pressure H at any point (x, y, z) is given by

$$H(x, y, z) = e^{-4xy} + \int_z^{x+y} e^{-t^2}\, \mathrm{d}t.$$

Let $P = (1, 1, 2)$ be a point on V.

(a) (5%) Find the equation of the tangent plane of V at P.

(b) (4%) Near P, the surface V defines implicitly $z = z(x, y)$ as a differentiable function in x and y. Find the directional derivative of $z(x, y)$ at P in the direction $\mathbf{u} = \langle -3, 4\rangle$.

(c) (5%) Find the linearization of the air pressure $H(x, y, z)$ at P.

(d) (4%) It is known that the temperature function $T(x, y, z)$, subject to a fixed air pressure $H(x, y, z) = e^{-4}$, attains its maximum value at P. Moreover, it is known that $T_y(P) = 25e^{-4}$. Find $\nabla T(P)$ and hence find the maximum rate of change of $T(x, y, z)$ at P.

Solution:

(a) (1M) Let $F(x, y, z) = z^2x^4 + zy^6 - x^2 - y^4$. Then
(1M+1M) $\nabla F(x, y, z) = \langle 4x^3z^2 - 2x,\ 6y^5z - 4y^3,\ 2zx^4 + y^6\rangle$. So $\nabla F(P) = \langle 14,\ 8,\ 5\rangle$.
(2M) Hence, the equation of tangent plane is given by

$$\langle 14,\ 8,\ 5\rangle \cdot \langle x - 1,\ y - 1,\ z - 2\rangle = 0$$

or equivalently $14x + 8y + 5z = 32$.

Grading scheme for Q2a

- (1M) Finding the function $F(x, y, z)$ that underlies the given level surface.
- (1M) Finding $\nabla F(x, y, z)$.
- (1M) Finding $\nabla F(P)$.
- (1M) Demonstrating knowledge of tangent plane
- (1M) Correct answer

Remarks.

(i) If a student makes minor calculation mistakes but have demonstrated sufficient knowledge about equation of tangent plane. 3-4M will be awarded, depending on how many components of ∇F are correctly computed.

(ii) At most 2M overall will be given to a candidate for very poor errors, such as not evaluating ∇F at P (hence yielding a 'non-linear' tangent 'plane').

(b) (1M+1M) By Implicit Function Theorem,

$$z_x(P) = -\frac{f_x(P)}{f_z(P)} = -\frac{14}{5} \text{ and } z_y(P) = -\frac{f_y(P)}{f_z(P)} = -\frac{8}{5}.$$

Normalizing $\langle -3,\ 4\rangle$ gives $\vec{u} = \langle -\frac{3}{5}, \frac{4}{5}\rangle$.
Therefore, the required directional derivative is

$$D_{\vec{u}}z(P) = \underbrace{\nabla z(P) \cdot \vec{u}}_{\text{1M}} = \underbrace{\frac{2}{5}}_{\text{1M}}.$$

Grading scheme for Q2b

- (1M) Knowing the statement of Implicit function theorem
- (1M) Finding $z_x(P)$ and $z_y(P)$ correctly
- (1M) Knowing the formula 'gradient dot direction'
- (1M) Correct answer

Remarks.

(i) Deduct 1M overall if a student forgets to normalize the given direction

(ii) If a student makes sign errors/minor calculation mistakes, he/she can earn at most 3M.

(c) (3M) The partial derivatives are

$$H_x(x,y,z) = -4ye^{-4xy} + e^{-(x+y)^2},\ H_y(x,y,z) = -4xe^{-4xy} + e^{-(x+y)^2},\ H_z(x,y,z) = -e^{-z^2}.$$

Therefore, $\nabla H(1,1,2) = \langle -3e^{-4},\ -3e^{-4},\ -e^{-4}\rangle$.
Thus, the linearization of H at $(1,1,2)$ is

$$\begin{aligned} L(x,y,z) &= \underbrace{H(1,1,2) + \nabla H(1,1,2) \cdot \langle x-1,\ y-1,\ z-2\rangle}_{1\text{M}} \\ &= \underbrace{e^{-4}}_{1\text{M}} - 3e^{-4}(x-1) - 3e^{-4}(y-1) - e^{-4}(z-2). \end{aligned}$$

Grading scheme for Q2c

- (3M) Finding $\nabla H(1,1,2)$. (Partial credits possible)
- (1M) Knowing what linearization is
- (1M) The value of $H(1,1,2)$ and the correct answer

Remarks.

(i) We reserve the rights to deduce marks for students who find $\nabla H(1,1,2)$ correctly by nonsensical argument.

(ii) Partial credits are possible for minor calculation errors in finding $\nabla H(1,1,2)$.

(d) (1M) By the method of Lagrange multipliers, we have $\nabla T(P) = \lambda \nabla H(P)$.
By comparing the y-component, we have $25e^{-4} = \lambda(-3e^{-4})$ so $\lambda = -\dfrac{25}{3}$.
(1M) Hence, $\nabla T(P) = \langle 25e^{-4},\ 25e^{-4},\ \dfrac{25}{3}e^{-4}\rangle$.
(1M) The maximum rate of change of T at P is $\|\nabla T(P)\|$,
(1M) which equals $\sqrt{(25e^{-4})^2 + (25e^{-4})^2 + (\dfrac{25}{3}e^{-4})^2}$.

Grading scheme for Q2d

- (1M) Correct approach (use Lagrange multipliers)
- (1M) Correct $\nabla T(P)$
- (1M) Knowing max. rate of change is given by the length of gradient
- (1M) Correct max. rate of change

Remarks.

(i) We can give up to 3M for students who obtained incorrect $\nabla H(P)$ from (c) but have done all subsequent calculations correctly.

3. Let $f(x, y) = 3x - x^2 + 4xy - 4y^2 - x^3$.

(a) (4%) Find all the critical points of $f(x, y)$.

(b) (6%) Determine whether each critical point in (a) is a local minimum, a local maximum, or a saddle.

Solution:

(a) Find all points (a, b) that satisfies $f_x(a, b) = 0$ and $f_y(a, b) = 0$.

$$f_x = 3 - 2x + 4y - 3x^2$$

$$f_y = 4x - 8y$$

From $f_y = 0$ we get $x = 2y$. Plug into $f_x = 0$ to get $3 - 3x^2 = 0$. Therefore $x = \pm 1$, and $y = \pm 0.5$, respectively.

The list of critical points includes $(1, 0.5)$ and $(-1, -0.5)$. $\square$

(b) Use the second partial derivatives test.

$$f_{xx} = -2 - 6x$$

$$f_{xy} = 4$$

$$f_{yy} = -8$$

At the point $(1, 0.5)$, $D = f_{xx}f_{yy} - (f_{xy})^2 = 64 - 16 > 0$. Along with $f_x x(1, 0.5) < 0$, we conclude that $(1, 0.5)$ is a local maximum.

At the point $(-1, -0.5)$, $D = f_{xx}f_{yy} - (f_{xy})^2 = -32 - 16 < 0$. We conclude that $(-1, -0.5)$ is a saddle. $\square$

Grading guideline:

(2%) for correct partial derivatives and knowing the definition of critical points.

(2%) for solving the points correctly. (all or nothing)

(2%) for understanding the second partial derivatives test and finding the second partials.

(2%) for checking D value at whatever critical points they found in (a).

(2%) for stating the correct conclusion based on their work from above.

4. (12%) To find the plane passing through $(1,2,3)$ that cuts off the smallest solid in the first octant, we need to

$$\text{minimize } V(a,b,c) = abc \text{ subject to the constraint } \frac{1}{a} + \frac{2}{b} + \frac{3}{c} = 1.$$

Solve the above optimization problem by the method of Lagrange multipliers.

Solution:

We want to minimize abc with the constraint $1/a + 2/b + 3/c = 1$. Remember that a, b, c are positive since they are the intercepts of the plane $x/a + y/b + z/c = 1$.

Denote $V = abc$ and $g = 1/a + 2/b + 3/c$. We want to minimize $V(a,b,c)$ with $g(a,b,c) = 1$. The absolute minimum must be a solution for the equation of the Lagrange multipliers. Let us solve grad $V = \lambda$ grad g and $g = 1$. The first equation gives

$$bc = \frac{-\lambda}{a^2}, ac = \frac{-2\lambda}{b^2}, ab = \frac{-3\lambda}{c^2}.\text{(5 points)}$$

We can deduce $b = 2a$ and $c = 3a$ (3 points); plug these identities in $1/a + 2/b + 3/c = 1$ to get $(a,b,c) = (3,6,9)$ (2 points). Since this is the only solution, it must be the absolute minimum, and hence the plane is $x/3 + y/6 + z/9 = 1$ (2 points).

5. Evaluate the following integrals

(a) (8%) $\displaystyle\int_0^1 \int_{\sin^{-1}(y)}^{\frac{\pi}{2}} \frac{\cos(x)}{2+\cos^2(x)}\,\mathrm{d}x\,\mathrm{d}y.$

(b) (8%) $\displaystyle\iint_D (x^2+4x+y^2)\,\mathrm{d}A$ where $D = \{(x,y)\in\mathbb{R}^2 : (x+2)^2+y^2 \le 4\}$.

Solution:

1. (2M) By drawing the region that underlies the given integral :

$$\{(x,y)\in\mathbb{R}^2 : 0\le y\le \sin x,\, 0\le x\le \frac{\pi}{2}\}.$$

Using Fubini's Theorem, we have

$$\begin{aligned}
\int_0^1 \int_{\arcsin y}^{\frac{\pi}{2}} \frac{\cos x}{2+\cos^2 x}\,dx\,dy &= \int_0^{\frac{\pi}{2}} \int_0^{\sin x} \frac{\cos x}{2+\cos^2 x}\,dy\,dx \qquad \cdots(3\text{M})\\
&= \int_0^{\frac{\pi}{2}} \frac{\sin x\cos x}{2+\cos^2 x}\,dx\\
&= \left[-\frac{1}{2}\ln(2+\cos^2 x)\right]_{x=0}^{x=\frac{\pi}{2}} \qquad \cdots(2\text{M})\\
&= -\frac{1}{2}(\ln 2-\ln 3). \qquad \cdots(1\text{M})
\end{aligned}$$

<u>Grading scheme for Q5a</u>

- (2M) For the correct sketch of the region
- (3M) For correctly switching the order of integration (for the four integration limits)
- (2M) For an anti-derivative of $\dfrac{\sin x\cos x}{2+\cos^2 x}$
- (1M) Correct answer

Remarks.

(i) No credits will be given to the whole question for irrational (and improbable) approaches or to those who do not have a clue on how Fubini's Theorem works.

(b) **<u>(Method 1 - using 'shifted' polar coordinates)</u>**

(1M) Let $x=-2+r\cos\theta$ and $y=r\sin\theta$.

(1M) D becomes a disk of radius 2, centred at $(0,0)$.

$$\begin{aligned}
\iint_D (x^2+4x+y^2)\,dA &= \underbrace{\int_0^{2\pi}\int_0^2}_{2\text{M}} \underbrace{(-4+r^2)}_{1\text{M}}\cdot \underbrace{r}_{1\text{M}}\,dr\,d\theta\\
&= 2\pi\cdot(-4)\\
&= -8\pi \cdots (2\text{M})
\end{aligned}$$

<u>(Method 2 - using 'ordinary' polar coordinates)</u>

(1M) Let $x=r\cos\theta$ and $y=r\sin\theta$.

(1M) Then the boundary of D becomes $r=-4\cos\theta$ where $-\dfrac{\pi}{2}\le\theta\le\dfrac{3\pi}{2}$.

$$\begin{aligned}
\iint_D (x^2+4x+y^2)\,dA &= \int_0^{\pi/2}\int_{-4\cos\theta}^0 (r^2+4r\cos\theta)\cdot(-r)\,dr\,d\theta + \int_{\pi/2}^{\pi}\int_0^{-4\cos\theta}(r^2+4r\cos\theta)\cdot r\,dr\,d\theta\\
&= \underbrace{\int_0^{\pi}\int_0^{-4\cos\theta}}_{2\text{M}} \underbrace{(r^2+4r\cos\theta)}_{1\text{M}}\cdot\underbrace{r}_{1\text{M}}\,dr\,d\theta\\
&= \int_0^{\pi} -\frac{64}{3}\cos^4\theta\,d\theta\\
&= -8\pi \cdots (2\text{M})
\end{aligned}$$

Grading scheme for Q5b

- (1M) For any reasonable substitution
- (1M) For a description of the region in the new coordinate system
- (2M) For the integration limits
- (1M) For the transformed integrand
- (1M) For Jacobian
- (2M) For correct answer

6. (10%) Let E be the lamina that occupies an elliptical region :

$$E = \{(x,y) \in \mathbb{R}^2 : x^2 + 2y^2 \le 4\}$$

Suppose the density function for E is $\rho(x,y) = x^2$. Find the total mass of E.

Solution:

(2 points) We consider $x = 2r\cos\theta$ and $y = \sqrt{2}r\sin\theta$.
Then $E = \{(r,\theta) : 0 \le r \le 1, 0 \le \theta \le 2\pi\}$ and

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} 2\cos\theta & -2r\sin\theta \\ \sqrt{2}\sin\theta & \sqrt{2}r\cos\theta \end{vmatrix} = 2\sqrt{2}r. \quad \text{(4 points)}$$

Thus the total mass of E is

$$\begin{aligned} M &= \iint_E \rho(x,y)\,\mathrm{d}A = \iint_E x^2\,\mathrm{d}A \quad \text{(1 point)} \\ &= \int_0^{2\pi}\int_0^1 (2r\cos\theta)^2 \cdot 2\sqrt{2}r\,\mathrm{d}r\,\mathrm{d}\theta \quad \text{(1 point)} \\ &= 8\sqrt{2}\int_0^{2\pi}\int_0^1 r^3\cos^2\theta\,\mathrm{d}r\,\mathrm{d}\theta \\ &= 8\sqrt{2}\int_0^{2\pi}\cos^2\theta\,\mathrm{d}\theta\int_0^1 r^3\,\mathrm{d}r \\ &= 2\sqrt{2}\pi. \quad \text{(2 points)} \end{aligned}$$

7. In this question, we want to find the volume integral $\iiint_S 1\,\mathrm{d}V$ of the following region :

$$S = \{(x,y,z) \in \mathbb{R}^3 \,:\, x \geq 0,\, y \geq 0,\, z \geq 0,\, \sqrt{x} + \sqrt{y} + z \leq 1\}$$

by making the change of variables $x = u^2$, $y = v^2$ and $z = w$.

(a) (3%) Compute the Jacobian $\dfrac{\partial(x,y,z)}{\partial(u,v,w)}$.

(b) (7%) Hence, find the volume of the region S.

Solution:

(a)

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix} \qquad \text{(+1 for the definition of Jacobian)}$$

$$= \begin{vmatrix} 2u & 0 & 0 \\ 0 & 2v & 0 \\ 0 & 0 & 1 \end{vmatrix} = 4uv \qquad \text{(+2 for partial derivatives and final answer)}$$

(b) If we restrict the transformation $T : x = u^2, y = v^2, z = w$ on the first octant, then the transformation is one-to-one and the corresponding region of S in the uvw space is the tetrahedron

$$D = \{(u,v,w) \mid u \geq 0, v \geq 0, w \geq 0, u + v + w \leq 1\} \qquad (+1)$$

Hence

$$V(S) = \iiint_S 1\; dV = \iiint_D 1 \cdot \left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| dwdvdu = \iiint_D 4uv\; dwdvdu \quad (+1)$$

Moreover, D can be described as a type I region:

$$D = \{(u,v,w) \mid 0 \leq u \leq 1,\; 0 \leq v \leq 1 - u,\; 0 \leq w \leq 1 - u - v\}$$

Therefore,

$$V(S) = \iiint_D 4uv\; dwdvdu = \int_0^1 \int_0^{1-u} \int_0^{1-u-v} 4uv\; dwdvdu \qquad (+3)$$

$$= \int_0^1 \int_0^{1-u} 4uv(1-u-v)\; dvdu = \int_0^1 \frac{2}{3}u(1-u)^3\; du = \frac{1}{30}. \qquad (+2)$$

Students can describe D as other types, for example,

$$D = \{(u,v,w) \mid 0 \leq w \leq 1,\; 0 \leq v \leq 1 - w,\; 0 \leq u \leq 1 - v - w\}.$$

Then

$$V(S) = \int_0^1 \int_0^{1-w} \int_0^{1-v-w} 4uv\; dudvdw = \int_0^1 \int_0^{1-w} 2(1-v-w)^2 v\; dvdw$$

$$= \int_0^1 \frac{1}{6}(w-1)^4\; dw = \frac{1}{30}.$$

8. Let $I = \iiint_H e^z \,\mathrm{d}V$ where $H = \{(x,y,z) \in \mathbb{R}^3 : x^2+y^2+z^2 \le 1, x \ge 0, y \ge 0, z \ge 0\}$.

(a) (5%) Express I as an iterated integral in *spherical coordinates* in the specified order of integration :

$$I = \int_a^b \int_c^d \int_e^f g(\rho,\varphi,\theta)\,\mathrm{d}\varphi\,\mathrm{d}\rho\,\mathrm{d}\theta.$$

(b) (5%) Hence, evaluate I.

Solution:

(a)

$$I = \int_0^{\frac{\pi}{2}} \int_0^1 \int_0^{\frac{\pi}{2}} e^{\rho\cos\phi}\rho^2 \sin\phi \; d\phi d\rho d\theta$$

Hence $g(\rho,\phi,\theta) = e^{\rho\cos\phi}\rho^2\sin\phi$ (+1 for $e^{\rho\cos\phi}$, +1 for Jacobian).
The range of ρ is $[0,1]$ (+1).
The range of ϕ is $[0,\frac{\pi}{2}]$ (+1).
The range of θ is $[0,\frac{\pi}{2}]$ (+1).
If students obtain correct ranges for ρ,ϕ,θ but write down the iterated integral in wrong order (for example, $d\rho\, d\phi\, d\theta$), then they loss 1 point.

(b)

$$I = \int_0^{\frac{\pi}{2}} \int_0^1 \int_0^{\frac{\pi}{2}} e^{\rho\cos\phi}\rho^2\sin\phi \; d\phi d\rho d\theta = \int_0^{\frac{\pi}{2}} \int_0^1 \left(-\rho e^{\rho\cos\phi}\right)\Big|_{\phi=0}^{\phi=\frac{\pi}{2}} \; d\rho d\theta \quad \text{(+2 for integrating w.r.t. } \phi)$$

$$= \int_0^{\frac{\pi}{2}} \int_0^1 -\rho + \rho e^{\rho} \; d\rho d\theta = \int_0^{\frac{\pi}{2}} \left(-\frac{\rho^2}{2} + \rho e^{\rho} - e^{\rho}\right)\Bigg|_{\rho=0}^{\rho=1} \; d\theta \qquad \text{(+2 for integrating w.r.t. } \rho)$$

$$= \frac{\pi}{4} \qquad \text{(+1 for the final answer)}$$