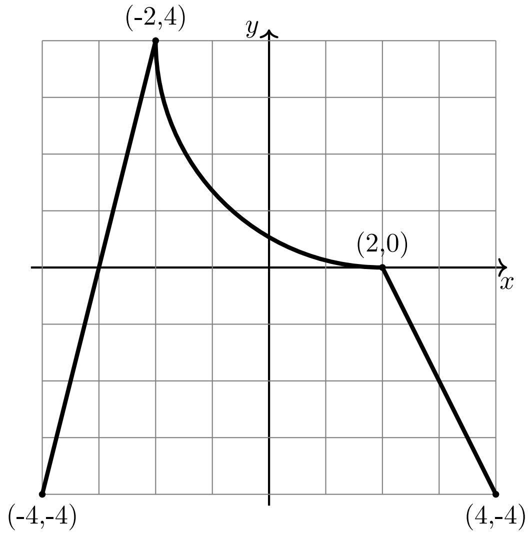
1. (10%) Fill in the blanks. 填空題。

The given graph $y = f(x)$ contains 2 line segments and 1 circular arc (a quarter circle).
底下是函數 $y = f(x)$ 的函數圖形，由兩條線段與四分之一圓弧組成。



Evaluate 計算

(a) $\int_{-2}^{2} f(x)\, dx =$ ________

(b) $\int_{2}^{4} f(x)\, dx =$ ________

(c) $\int_{-4}^{4} f(x)\, dx =$ ________

(d) Let $g(x) = \int_{-4}^{x} f(t)\, dt$. The function $g(x)$ is increasing on the interval
定義 $g(x) = \int_{-4}^{x} f(t)\, dt$。試求出函數 g 遞增的區間。(________ , ________)

(e) The degree two Taylor polynomial of the function $F(x) = \sqrt{4+x}$ at $x = 0$ is
函數 $F(x) = \sqrt{4+x}$ 在 $x = 0$ 的二次泰勒多項式是

$$P_2(x) = \underline{\qquad\qquad} + \underline{\qquad\qquad}\ x + \underline{\qquad\qquad}\ x^2$$

(f) Suppose we observe that a population $P(t)$ is proportional to the square of the growth rate $\frac{dP}{dt}$, where t is a time variable. Express the model as a differential equation.
已知某人口模型中，"人口函數 $P(t)$" 和 "人口成長率 $\frac{dP}{dt}$ 的平方" 成正比，底下哪個微分方程是這個模型的方程式？

________ (A) $\frac{dP}{dt} = kt^2$ (B) $\frac{dP}{dt} = kP^2$ (C) $\frac{dP}{dt} = k\sqrt{P}$ (D) $\frac{dP}{dt} = k\sqrt{t}$

Solution:

Grading scheme of Question 1 Answers only. No partial credits.
(a) $16 - 4\pi$ (2%)
(b) -4 (2%)
(c) $12 - 4\pi$ (1%)
(d) $(-3, 2)$ (1%)
(e) $2 + \frac{1}{4}x - \frac{1}{64}x^2$ (1+1+1%)
(f) C (1%)

2. (10%) Let $F(x) = \int_{-\sqrt{x}}^{\sqrt{x}} e^{-t^2}\,dt$. 定義 $F(x) = \int_{-\sqrt{x}}^{\sqrt{x}} e^{-t^2}\,dt$。

(a) Use the Fundamental Theorem of Calculus to find $F'(x)$. 用微積分基本定理來計算 $F'(x)$。

(b) Evaluate the limit $\lim_{x\to 0^+} \frac{F(x)}{\sqrt{x}}$ with l'Hôpital's Rule. 用羅必達法則來計算極限 $\lim_{x\to 0^+} \frac{F(x)}{\sqrt{x}}$

Solution:

(a) Since e^{-t^2} is continuous on $\mathbb{R}$ and

$$F(x) = \int_0^{\sqrt{x}} e^{-t^2}\,dt - \int_0^{-\sqrt{x}} e^{-t^2}\,dt, \text{(1\%)}$$

the Fundamental Theorem of Calculus implies

$$F'(x) = \underbrace{e^{-(\sqrt{x})^2}}_{(1\%)} \cdot \underbrace{\frac{1}{2\sqrt{x}}}_{(1\%)} - \underbrace{e^{-(-\sqrt{x})^2}}_{(1\%)} \cdot \underbrace{\left(-\frac{1}{2\sqrt{x}}\right)}_{(1\%)} = \frac{e^{-x}}{\sqrt{x}}$$

Remark. Students cannot quote the formula for

$$\frac{d}{dx}\left(\int_{g(x)}^{h(x)} F(t)\,dt\right) = F(h(x))h'(x) - F(g(x))g'(x)$$

directly unless they give its derivation (correctly).

(b) The given limit is of $\frac{0}{0}$ form. (1%)
By L'Hospital's rule, we have

$$\lim_{x\to 0^+} \frac{F(x)}{\sqrt{x}} = \underbrace{\lim_{x\to 0^+} \frac{F'(x)}{\frac{1}{2\sqrt{x}}}}_{(1\%)} = \underbrace{\lim_{x\to 0^+} \frac{\frac{e^{-x}}{\sqrt{x}}}{\frac{1}{2\sqrt{x}}}}_{(1\%)} = \underbrace{\lim_{x\to 0^+} \frac{e^{-x}}{2}}_{(1\%)} = \underbrace{\frac{1}{2}}_{(1\%)}$$

Remark. All answers need to be fully and correctly justified.

3. (20%) Evaluate. 計算題。

(a) $\int_0^{\pi/4} \tan^3\theta \ \sec^2\theta \ d\theta$

(b) $\int_0^1 \frac{1}{x^2+2x+2} \ dx$

Solution:

(a)

$$\int_0^{\pi/4} \tan^3\theta \ \sec^2\theta \ d\theta = \int_0^{\pi/4} (\tan\theta)^3 \ d(\tan\theta) = \int_0^1 u^3 \ du = \left.\frac{u^4}{4}\right|_0^1 = \frac{1}{4}$$

(b)

$$\int_0^1 \frac{1}{x^2+2x+2} \ dx = \int_0^1 \frac{1}{(x+1)^2+1} \ d(x+1) = \int_1^2 \frac{1}{u^2+1} \ du = \tan^{-1} u\Big|_1^2 = \tan^{-1}(2) - \frac{\pi}{4}$$

Grading:

There are many ways for the student to get the correct answer. Read their work, -2% for each minor mistake. -3% for any antiderivative mistake.

If the student did not arrive at the correct answer (unfinished or major mistake), they get +3% for the first correct integration technique and +2% for the first correct antiderivative.

If the student mis-copied the problem, determine if the new integral is of similar difficulty. Grade normally if it is, otherwise max 5%.

4. (10%) Use the method of partial fractions to evaluate $\int \frac{x-1}{x^2(x^2+1)}\,dx$. Make sure to clearly express the rational function in its partial fraction form.

用部分分式積分法來計算 $\int \frac{x-1}{x^2(x^2+1)}\,dx$。

Solution:

Partial fractions:

$$\frac{x-1}{x^2(x^2+1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{Cx+D}{x^2+1} = \frac{A(x^3+x)+B(x^2+1)+Cx^3+Dx^2}{x^2(x^2+1)}$$

Constant term: $B = -1$.

Linear term: $A = 1$.

Other terms have zero coefficient: $C = -1, D = 1$.

Hence

$$\frac{x-1}{x^2(x^2+1)} = \frac{1}{x} - \frac{1}{x^2} - \frac{x}{x^2+1} + \frac{1}{x^2+1}$$

$$\int \frac{x-1}{x^2(x^2+1)}\,dx = \int \left(\frac{1}{x} - \frac{1}{x^2} - \frac{x}{x^2+1} + \frac{1}{x^2+1}\right)dx = \ln|x| + \frac{1}{x} - \frac{1}{2}\ln(x^2+1) + \tan^{-1}x + K$$

□

Grading:

4% for the partial fraction form. Equivalent forms are okay.

2% for solving the constants. Any minor error is -1%. If there are fewer than 4 constants they immediately get -1%.

4% for the antiderivative. If 1 constant is zero due to error above, -0%. If 2 or more constants are zero then -1% for each. Missing absolute value or missing $+K$ (using $+C$ is fine) is -1% each.

If the student miss a term in the first step, then it will be -3% due to losing 1% in each part.

Other methods to solve the problem will have max 8%, but gets -2% for each minor error and -4% for each major error.

5. (10%) Evaluate the improper integral $\int_0^4 \frac{dx}{4x+\sqrt{x}}$. 計算瑕積分 $\int_0^4 \frac{dx}{4x+\sqrt{x}}$。

Solution:

Let $u = \sqrt{x}$ (1%), so $du = \frac{1}{2\sqrt{x}}dx$ (1%)i.e. $dx = 2udu$ (2%). Thus, we have

$$\begin{aligned}\int \frac{dx}{4x+\sqrt{x}} &= \int \frac{2udu}{4u^2+u}(2\%)\\ &= \int \frac{2}{4u+1}du\\ &= \frac{2}{4}\ln|4u+1|\end{aligned}$$

(2% for correct answer. 1% if see any sign of ln). Finally, we have

$$\int_0^4 \frac{dx}{4x+\sqrt{x}} = \frac{2}{4}\ln|4\sqrt{x}+1|\Big|_0^4 = \frac{1}{2}\ln 9 = \ln 3$$

(WITHOUT CONSIDERING THE PREVIEWS INTEGRAL IS CORRECT OR NOT. 2% if the student reverses the process of substitute and plugin 4 and 0 to do subtraction (You don't need to check if the student compute the result right or wrong, as long as the student carry out the process correctly). 1% if the student only reverse the process of substitute or only do substitution.)

6. (8%) Given $e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n$. Let $g(x) = \int_0^x e^{t^2}\, dt$. Write down the Taylor series of $g(x)$ at $x = 0$.

定義 $g(x) = \int_0^x e^{t^2}\, dt$。已知 $e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n$，試求函數 $g(x)$ 在 $x = 0$ 的泰勒級數。

Solution:

$$\begin{aligned} g(x) &= \int_0^x \sum_{n=0}^{\infty} \frac{1}{n!} t^{2n} dt \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} \int_0^x t^{2n} dt \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} \frac{x^{2n+1}}{2n+1} \end{aligned}$$

(2% if you see t has power $2n$ before integral; 2% if you see t has power $2n+1$ after integral. 2% if you see $\frac{1}{2n+1}$ after integral. 2% if you see any sign that the student interchange the operation of summation and integration. -1% for any stupid mistake up to -2%.)

7. (12%) Given $\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} x^n$. Let $f(x) = x^2 \ln(1-x^3)$.

定義 $f(x) = x^2 \ln(1-x^3)$。已知 $\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} x^n$。

(a) Write down the Taylor series of $f(x)$ at $x = 0$. 求函數 $f(x)$ 在 $x = 0$ 的泰勒級數。

(b) Find $f^{(11)}(0)$. 用泰勒級數來計算 $f^{(11)}(0)$。

Solution:

(a) $f(x) = \sum_{n=1}^{\infty} \frac{-1}{n} x^{3n+2}$. (It's okay if you see -1^{2n+1}. 6% for this question, and mimic the grading scheme of Question 6).

(b) Set $11 = 3n + 2$ (2%). Solve $n = 3$ (1%). $f^{11}(0) = \frac{-11!}{3}$ (3%). (As long as the student use his/her result in the part (a), the student gets points)

8. (20%) Solve. 解微分方程。

(a) $\dfrac{dy}{dx} = 1 + y^2$

(b) $\dfrac{dy}{dx} - 2xy = xe^{x^2}\sin(2x)$ with initial value 初始條件 $y(0) = 1$.

Solution:

(a) By separating the variable,

$$\underbrace{\int \frac{1}{1+y^2}dy}_{(1\%)} = \underbrace{\int 1dx}_{(1\%)} \Rightarrow \underbrace{\tan^{-1}(y)}_{(2\%)} = \underbrace{x}_{(1\%)} + \underbrace{C}_{(1\%)}$$

Remark. Accept any equivalent answers.

(b) An integrating factor is given by $e^{\int(-2x)dx} = e^{-x^2}$ (2%). By multiplying it to the both sides,

$$\begin{aligned} e^{-x^2}\frac{dy}{dx} - 2xe^{-x^2}y &= x\sin(2x) \\ \Longrightarrow \quad \left(e^{-x^2}\cdot y\right)' &= x\sin(2x) \\ \Longrightarrow \quad e^{-x^2}\cdot y &= \int x\sin(2x)\,dx (3\%) \\ &= -\frac{x\cos(2x)}{2} + \int \frac{\cos(2x)}{2}dx (3\%) \\ &= -\frac{x\cos(2x)}{2} + \frac{\sin(2x)}{4} + C (2\%) \end{aligned}$$

Hence, $y = e^{x^2}\left(-\dfrac{x\cos(2x)}{2} + \dfrac{\sin(2x)}{4} + C\right)$.

Finally, we are given that when $x = 0$, $y = 1$,(1%)

$$1 = 1\cdot(0+0+C)$$

so $C = 1$(2%). From this we conclude that

$$y = e^{x^2}\left(-\frac{x\cos(2x)}{2} + \frac{\sin(2x)}{4} + 1\right).(1\%)$$

Remark. For any components that are worth $\geq 2\%$, partial credits are available for any minor calculation errors.