1. (10%) Let V be the subspace of $\mathbb{R}^4$, $V = \{(x, y, z, w) \in \mathbb{R}^4 : x - y - z = 0, y - z + w = 0, 2x + y - 5z + 3w = 0\}$.
   (a) (7%) Find the dimension and a basis of V.
   (b) (3%) Is $(1, -1, -1, 0)$ orthogonal to every vector in V?

Solution:

(a)

Solve the system of equations

$$\begin{cases} x - y - z = 0, \\ y - z + w = 0, \\ 2x + y - 5z + 3w = 0. \end{cases}$$

Convert into matrix

$$\begin{pmatrix} 1 & -1 & -1 & 0 \\ 0 & 1 & -1 & 1 \\ 2 & 1 & -5 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Reduce the matrix to obtain

$$\begin{pmatrix} 1 & 0 & -2 & 1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

The dimension of V is two. We can find a basis by setting z and w as our variables. So $x = 2z - w$ and $y = z - w$. In vector form the basis is

$$\begin{pmatrix} 2 \\ 1 \\ 1 \\ 0 \end{pmatrix} \text{ and } \begin{pmatrix} -1 \\ -1 \\ 0 \\ 1 \end{pmatrix}$$

□

(b)

The answer is yes because $x - y - z = 0$, which is the inner product of $(1, -1, -1, 0)$ and (x, y, z, w), from part (a). □

Grading:

- (a) Just like quiz 1, because linear algebra statements have many different ways to be phrased, make sure to read the student's work. (-1%) for any step that is unclear or confusing. (-2%) for any conceptual mistakes.
- (b) All or nothing. They must have an explanation to get points.

2. (10%) Consider $f(x, y, z) = 2x^2 + 3y^2 + 3z^2 + 4xy - 2yz$.

(a) (3%) Write down its associated symmetric matrix Q.

(b) (5%) Determine the definitness of Q.

(c) (2%) Is the origin (0,0,0) a local maximum, local minimum, or a saddle point of $f(x)$?

Solution:

(a) $Q = \begin{pmatrix} 2 & 2 & 0 \\ 2 & 3 & -1 \\ 0 & -1 & 3 \end{pmatrix}$. (Each of $a_{11}, a_{12}, a_{13}, a_{22}, a_{23}, a_{33}$ counts for 0.5 point.)

(b) Sol 1: $LPM_1 = 2 > 0$, $LPM_2 = \begin{vmatrix} 2 & 2 \\ 2 & 3 \end{vmatrix} = 2 > 0$, $LPM_3 = \begin{vmatrix} 2 & 2 & 0 \\ 2 & 3 & -1 \\ 0 & -1 & 3 \end{vmatrix} = 4 > 0$

Because LPM_1, LPM_2, LPM_3 are all positive, Q is positive definite by Sylvester Criterion.
(1 point for LPM_1, 1 point for LPM_2, 1 point for LPM_3, 1 point for positive definite, 1 point for correct reasoning.
If Students have wrong LPM's but they conclude definitness by correct reasoning, they get partial credits.)

Sol 2:

$$\begin{vmatrix} 2-\lambda & 2 & 0 \\ 2 & 3-\lambda & -1 \\ 0 & -1 & 3-\lambda \end{vmatrix} = (2-\lambda)[(3-\lambda)^2 - 1] - 2(6-2\lambda) = (2-\lambda)(\lambda^2 - 6\lambda + 8) + 4\lambda - 12 = -\lambda^3 + 8\lambda^2 - 16\lambda + 4,$$

$\lambda \approx 4.903, 2.806, 0.291$.

Because all eigenvalues are positive, Q is positive definite. (2 points for $\det(Q - \lambda I)$. 2 points for computing eigenvalues, 1 point for positive definite.)

(c) Because Q is positive definite, $f(x, y, z) = (x, y, z)Q\begin{pmatrix} x \\ y \\ z \end{pmatrix} > 0 = f(0, 0, 0)$.

Hence $f(0, 0, 0)$ is a local minimum. (2 points)

3. (24%) You want to have lunch containing rice, vegetables and meat with x, y, z units respectively. Your utility function is $U(x,y,z) = \frac{1}{4}\ln x + \frac{1}{4}\ln y + \frac{1}{2}\ln z$. Because you have a budget constraint and you are on a diet, you are subject to $\begin{cases} x + 2y + 5z \le 20 \text{ (budget constraint)} \\ x \le 4 \text{ (less rice)} \\ y \ge 3 \text{ (more vegetables)} \end{cases}$. Try to maximize your utility.

   (a) (4%) Check whether NDCQ is satisfied.

   (b) (2%) Write down the Lagrangian function.

   (c) (6%) Write down the complete first order conditions.

   (d) (7%) Find the maximum utility.

   (e) (5%) Estimate the maximum utility if you are on an easier diet, $x \le 4.1$, $y \ge 2.8$.

Solution:

(a) Let $g_1(x,y,z) = x + 2y + 5z$, $g_2(x,y,z) = x$, $g_3(x,y,z) = -y$.
Then constraints are $g_1(x,y,z) \le 20$, $g_2(x,y,z) \le 4$, $g_3 \le -3$.
$\nabla g_1 = (1,2,5)$, $\nabla g_2 = (1,0,0)$, $\nabla g_3 = (0,-1,0)$. (2 points for ∇g_1, ∇g_2, ∇g_3)
$\nabla g_1 \ne \mathbf{0}$, $\nabla g_2 \ne \mathbf{0}$, and $\nabla g_3 \ne \mathbf{0}$.
Any two of $\{\nabla g_1, \nabla g_2, \nabla g_3\}$ are linearly independent. ∇g_1, ∇g_2 and ∇g_3 are linearly independent.
Hence no matter what constraints are binding, NDCQ is always satisfied. (2 points for arguing that NDCQ is satisfied.)

(b) $L(x,y,z,\lambda_1,\lambda_2,\lambda_3) = \frac{1}{4}\ln x + \frac{1}{4}\ln y + \frac{1}{2}\ln z - \lambda_1(x + 2y + 5z - 20) - \lambda_2(x-4) - \lambda_3(-y+3)$.
(2 points. The answer $L = U - \lambda_1(x + 2y + 5z - 20) - \lambda_2(x-4) - \lambda_3(y-3)$ gets 1 point.)

(c)

$$
\begin{aligned}
L_x &= \frac{1}{4x} - \lambda_1 - \lambda_2 = 0 && (1)(1 \text{ point}) \\
L_y &= \frac{1}{4y} - 2\lambda_1 + \lambda_3 = 0 && (2)(1 \text{ point}) \\
L_z &= \frac{1}{2z} - 5\lambda_1 = 0 && (3)(1 \text{ point}) \\
&\lambda_1(x + 2y + 5z - 20) = 0 && (4)(0.5 \text{ point}) \\
&\lambda_2(x-4) = 0 && (5)(0.5 \text{ point}) \\
&\lambda_3(-y+3) = 0 && (6)(0.5 \text{ point}) \\
&x + 2y + 5z \le 20, x \le 4, y \ge 3 && (7)(0.5 \text{ point}) \\
&\lambda_1 \ge 0, \lambda_2 \ge 0, \lambda_3 \ge 0 && (8)(1 \text{ point})
\end{aligned}
$$

(d) Solve FOC.
(3) $\Rightarrow \lambda_1 = \frac{1}{10z} > 0$. Hence (4) $\Rightarrow x + 2y + 5z = 20$. (1 point)
From (8) $\lambda_2 \ge 0$, $\lambda_3 \ge 0$, we discuss the following 4 cases.

Case $\lambda_2 = \lambda_3 = 0$
Then (1)(2)(3) $\Rightarrow \frac{1}{4x} = \lambda_1,\ \frac{1}{4y} = 2\lambda_1,\ \frac{1}{2z} = 5\lambda_1 \Rightarrow x = \frac{1}{4\lambda_1},\ y = \frac{1}{8\lambda_1},\ z = \frac{1}{10\lambda_1}$
$x+2y+5z = \frac{1}{4\lambda_1} + \frac{1}{4\lambda_1} + \frac{1}{2\lambda_1} = 20 \Rightarrow \frac{1}{\lambda_1} = 20,\ x = 5,\ y = \frac{5}{2}$ which is contradict to $x \le 4$ and $y \ge 3$. (1 point)

Case $\lambda_2 = 0$, $\lambda_3 > 0$
Then (6) $\Rightarrow y = 3$. (1)(3) $\Rightarrow \frac{1}{4x} = \lambda_1,\ \frac{1}{2z} = 5\lambda_1$
$\Rightarrow x = \frac{1}{4\lambda_1},\ z = \frac{1}{10\lambda_1}$ and $x + 2y + 5z = \frac{1}{4\lambda_1} + 6 + \frac{1}{2\lambda_1} = 20$
$\Rightarrow \frac{3}{4}\frac{1}{\lambda_1} = 14 \Rightarrow \frac{1}{\lambda_1} = \frac{56}{3},\ x = \frac{1}{4\lambda_1} = \frac{14}{3} > 4$ contradiction! (1 point)

Case $\lambda_2 > 0$, $\lambda_3 = 0$
Then (5) $\Rightarrow x = 4$, (2)(3) $\Rightarrow \frac{1}{4y} = 2\lambda_1,\ \frac{1}{2z} = 5\lambda_1$. $\Rightarrow y = \frac{1}{8\lambda_1},\ z = \frac{1}{10\lambda_1}$ and $x+2y+5z = 4 + \frac{1}{4\lambda_1} + \frac{1}{2\lambda_1} = 20$

$\Rightarrow \frac{3}{4}\frac{1}{\lambda_1} = 16 \Rightarrow \frac{1}{\lambda_1} = \frac{64}{3}$ and $y = \frac{8}{3} < 3$. which is a contradiction. (1 point)

Case $\lambda_2 > 0$, $\lambda_3 > 0$
(5)(6) $\Rightarrow x = 4$, $y = 3$. Then $x + 2y + 5z = 20 \Rightarrow z = 2$

$$\begin{cases} (1) \Rightarrow \frac{1}{16} = \lambda_1 + \lambda_2 \\ (2) \Rightarrow \frac{1}{12} = 2\lambda_1 - \lambda_3 \\ (3) \Rightarrow \frac{1}{4} = 5\lambda_1 \end{cases} \quad \text{Hence } \lambda_1 = \frac{1}{20},\ \lambda_2 = \frac{1}{16} - \frac{1}{20} = \frac{1}{80},\ \lambda_3 = \frac{1}{10} - \frac{1}{12} = \frac{1}{60}.$$

The solution is $(x, y, z, \lambda_1, \lambda_2, \lambda_3) = (4, 3, 2, \frac{1}{20}, \frac{1}{80}, \frac{1}{60})$.
The maximum ultility is $\frac{1}{4}\ln 4 + \frac{1}{4}\ln 3 + \frac{1}{2}\ln 2 = \ln 2 + \frac{1}{4}\ln 3$.
(1 point for $(x, y, z) = (4, 3, 2)$, 1 point for $(\lambda_1, \lambda_2, \lambda_3) = (\frac{1}{20}, \frac{1}{80}, \frac{1}{60})$, 1 point for the maximum ultility $\ln 2 + \frac{1}{4}\ln 3$.)

(e) Consider the optimization problem. Maxize $\frac{1}{4}\ln x + \frac{1}{4}\ln y + \frac{1}{2}\ln z$ subject to $x + 2y + 5z \le 20$, $x \le a$, $y \ge b$.
The Lagrangian is

$$L = \frac{1}{4}\ln x + \frac{1}{4}\ln y + \frac{1}{2}\ln z - \lambda_1(x + 2y + 5z - 20) - \lambda_2(x - a) - \lambda_3(-y + b). \quad \text{(1 point)}$$

Let the maximum ultility be $U_{max}(a, b)$.
Then $U_{max}(4, 3) = \ln 2 + \frac{1}{4}\ln 3$.
$\frac{\partial U_{max}}{\partial a}\Big|_{(4,3)} = \lambda_2\Big|_{(4,3)} = \frac{1}{80}$, $\frac{\partial U_{max}}{\partial b}\Big|_{(4,3)} = -\lambda_3\Big|_{(4,3)} = -\frac{1}{60}$. (1 point for $\frac{\partial U_{max}}{\partial a}$, 1 point for $\frac{\partial U_{max}}{\partial b}$)
Hence $U_{max}(4.1, 2.8) \approx U_{max}(4, 3) + \frac{\partial U_{max}}{\partial a} \times (0.1) + \frac{\partial U_{max}}{\partial b} \times (-0.2) = \ln 2 + \frac{1}{4}\ln 3 + \frac{1}{800} + \frac{1}{300}$
(1 point for linear approximation, 1 point for final answer.)

4. (18%) Consider the optimization problem:

$$\text{Maximize } f(x,y,z) = xyz \text{ subject to } x + y + z^2 \le 5,\ x \ge 0,\ y \ge 0,\ z \ge 0.$$

(a) (4%) Verify that Kuhn-Tucker's NDCQ is satisfied for this problem.

(b) (2%) Write down the Kuhn-Tucker's Lagrangian.

(c) (6%) Write down the complete first order conditions in Kuhn-Tucker's formulation.

(d) (6%) Find solutions of the first order conditions such that $x > 0$, $y > 0$, $z > 0$.

Solution:

(a) Suppose that $x+y+z^2 \le 5$ is binding. (1%) The full Jacobian matrix is $\begin{pmatrix} 1 & 1 & 2z \end{pmatrix}$. (1%). For $x, y, z \ne 0$. the full Jacobian matrix is $\begin{pmatrix} 1 & 1 & 2z \end{pmatrix}$. For $x = 0, y, z \ne 0$ and $y = 0, x, z \ne 0$, the reduced Jacobian matrix is $\begin{pmatrix} 1 & 2z \end{pmatrix}$. For $z = 0, x, y \ne 0$, the reduced Jacobian matrix is $\begin{pmatrix} 1 & 1 \end{pmatrix}$. For $x = z = 0, y \ne 0$ and $y = z = 0, x \ne 0$, the reduced Jacobian matrix is $\begin{pmatrix} 1 \end{pmatrix}$. For $x = y = 0$, then $z = 5$ and the reduced Jacobian matrix is $\begin{pmatrix} 10 \end{pmatrix}$. All the case is rank 1. So Kuhn-Tucker's NDCQ is satisfied (2%).

(b) The Kuhn-Tucker' s Lagrangian is given by

$$\widetilde{L}(x,y,z,\lambda) = xyz - \lambda(x + y + z^2 - 5). \ (2\%)$$

(c) The FOCs are

$$x\widetilde{L}_x = x(yz - \lambda) = 0 \tag{1}$$
$$y\widetilde{L}_y = y(xz - \lambda) = 0 \tag{2}$$
$$z\widetilde{L}_z = z(xy - 2\lambda z) = 0 \tag{3}$$
$$\lambda\widetilde{L}_\lambda = -\lambda(x + y + z^2 - 5) = 0 \tag{4}$$
$$\widetilde{L}_x = yz - \lambda \le 0,\ \widetilde{L}_y = xz - \lambda \le 0,\ \widetilde{L}_z = xy - 2\lambda z \le 0,\ \widetilde{L}_\lambda = -(x + y + z^2 - 5) \ge 0 \tag{5}$$
$$x \ge 0,\ y \ge 0,\ z \ge 0,\ \lambda \ge 0. \tag{6}$$

Grading: 0.5% for each equality and inequality.

(c) From (1)-(3) and $x, y, z > 0$, we have

$$\lambda = yz = xz = \frac{xy}{2z} \ \Rightarrow\ x = y = 2z^2 \text{ and } \lambda > 0. \ (2\%) \tag{7}$$

By (4) and (7), we have $5z^2 = 5$. So $z = 1$ by (6). (2%) Then $x = y = 2$ (1%) and $\lambda = 1 \cdot 2 = 2$ (1%) by (7). So the solution $(x, y, z, \lambda) = (2, 2, 1, 2)$.

5. (12%) Find extreme values of the function $f(x,y,z) = 6 - x^2 - 2y^2 - 2z^2$ subject to $x + y = 1$ and $-y + 2z = 2$.

(a) (4%) Suppose that $(x, y, z, \mu_1, \mu_2) = (1, 0, 1, \mu_1^*, \mu_2^*)$ is a critical point of the Lagrangian function

$$L(x, y, z, \mu_1, \mu_2) = 6 - x^2 - 2y^2 - 2z^2 - \mu_1(x + y - 1) - \mu_2(-y + 2z - 2).$$

Find μ_1^*, μ_2^*.

(b) (4%) Write down the bordered Hessian matrix at $(x, y, z, \mu_1, \mu_2) = (1, 0, 1, \mu_1^*, \mu_2^*)$.

(c) (4%) Determine whether $f(1, 0, 1)$ is a local maximum, local minimum or neither on the constraint set by second order conditions.

Solution:

(a) We compute that $L_x = -2x - \mu_1$, (0.5%) $L_y = -4y - \mu_1 + \mu_2$, (0.5%) $L_z = -4z - 2\mu_2$, (0.5%) $L_{\mu_1} = -(x + y - 1)$ (0.5%) and $L_{\mu_2} = -(-y + 2z - 2)$ (0.5%). Since $(1, 0, 1, \mu_1^*, \mu_2^*)$ is a critical point of L, we have $L_x = -2 - \mu_1^* = 0$ $L_y = -4(0) - \mu_1^* + \mu_2^* = 0$, $L_z = -4 - 2\mu_2^* = 0$, $L_{\mu_1} = -(1 + 0 - 1) = 0$ and $L_{\mu_2} = -(-0 + 2(1) - 2) = 0$. (1%) So $\mu_1^* = -2$ and $\mu_2^* = -2$. (0.5%)

(b) Let $g_1(x, y, z) = x + y$ and $g_2(x, y, z) = -y + 2z$. Then the bordered Hessian matrix at $(x, y, z, \mu_1, \mu_2) = (1, 0, 1, -2, -2)$ is

$$\left.\begin{pmatrix} 0 & 0 & g_{1x} & g_{1y} & g_{1z} \\ 0 & 0 & g_{2x} & g_{2y} & g_{2z} \\ g_{1x} & g_{2x} & L_{xx} & L_{xy} & L_{xz} \\ g_{1y} & g_{2y} & L_{xy} & L_{yy} & L_{yz} \\ g_{1z} & g_{2z} & L_{xz} & L_{yz} & L_{zz} \end{pmatrix}\right|_{(x,y,z,\mu_1,\mu_2)=(1,0,1,-2,-2)} \quad (2\%) = \begin{pmatrix} 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & -1 & 2 \\ 1 & 0 & -2 & 0 & 0 \\ 1 & -1 & 0 & -4 & 0 \\ 0 & 2 & 0 & 0 & -4 \end{pmatrix}. (2\%)$$

(c) In this problem, there are three variables and two constraints. So we consider the last 3-2 LPM of the bordered Hessian matrix. (1%) Then

$$\text{LPM}_5 = \begin{vmatrix} 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & -1 & 2 \\ 1 & 0 & -2 & 0 & 0 \\ 1 & -1 & 0 & -4 & 0 \\ 0 & 2 & 0 & 0 & -4 \end{vmatrix} = \begin{vmatrix} 0 & 0 & -1 & 2 \\ 1 & 0 & 0 & 0 \\ 1 & -1 & -4 & 0 \\ 0 & 2 & 0 & -4 \end{vmatrix} - \begin{vmatrix} 0 & 0 & 0 & 2 \\ 1 & 0 & -2 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 2 & 0 & 4 \end{vmatrix}$$

$$= -\begin{vmatrix} 1 & 0 & 0 \\ 1 & -1 & 0 \\ 0 & 2 & -4 \end{vmatrix} - 2\begin{vmatrix} 1 & 0 & 0 \\ 1 & -1 & -4 \\ 0 & 2 & 0 \end{vmatrix} + 2\begin{vmatrix} 1 & 0 & -2 \\ 1 & -1 & 0 \\ 0 & 2 & 0 \end{vmatrix} = -4 - 2(8) + 2(-4) = -28. \ (2\%)$$

Since LPM_5 has the same sign with $(-1)^3 = -1$. Hence, the SOC implies that the given point is a local maximum. (1%)

6. (26%) We want to maximize $f(x,y,z) = 2z^2 + 2xy - 2xz - 2yz$ under the constraint $x^2 + y^2 + z^2 = 1$. The Lagrangian function for this optimization problem is

$$L(x,y,z,\mu) = 2z^2 + 2xy - 2xz - 2yz - \mu(x^2 + y^2 + z^2 - 1).$$

(a) (3%) Suppose that $(x,y,z,\mu) = (\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \mu^*)$ is a solution of the first order conditions. Find μ^*.

(b) (7%) Write down the bordered Hessian matrix at $(x,y,z,\mu) = (\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \mu^*)$. Then determine whether $f(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}})$ is a local maximum, local minimum, or neither on the constraint set by second order conditions.

Suppose that (x^*, y^*, z^*, μ^*) is a solution of the first order conditions. Then we can show that $\mathbf{v} = \begin{pmatrix} x^* \\ y^* \\ z^* \end{pmatrix}$ is an eigenvector of $A = \begin{pmatrix} 0 & 1 & -1 \\ 1 & 0 & -1 \\ -1 & -1 & 2 \end{pmatrix}$ with eigenvalue μ^*.

(c) (9%) Find all eigenvalues and corresponding unit eigenvectors(eigenvectors with length 1) of $A = \begin{pmatrix} 0 & 1 & -1 \\ 1 & 0 & -1 \\ -1 & -1 & 2 \end{pmatrix}$.

(d) (2%) Find the maximum value of f on $x^2 + y^2 + z^2 = 1$.

(e) (5%) Estimate the maximum value of $0.03x^2 + 2z^2 + 2xy - 2xz - 2yz$ under the constraint $x^2 + y^2 + z^2 = 1.02$.

Solution:

(a)

First order conditions:

$$\begin{cases} 2y - 2z - 2\mu x = 0 \\ 2x - 2z - 2\mu y = 0 \\ 4z - 2x - 2y - 2\mu z = 0 \\ x^2 + y^2 + z^2 = 1 \end{cases}$$

By plugging in, we get $\mu^* = 0$. □

(b)

The bordered Hessian is a 4×4 matrix. We need

$$L_{xx} = -2\mu, L_{xy} = 2, L_{xz} = -2, L_{yy} = -2\mu, L_{yz} = -2, L_{zz} = 4 - 2\mu$$

The bordered Hessian (with x^*, y^*, z^*, μ^* plugged in) is

$$\begin{pmatrix} 0 & \frac{2}{\sqrt{3}} & \frac{2}{\sqrt{3}} & \frac{2}{\sqrt{3}} \\ \frac{2}{\sqrt{3}} & 0 & 2 & -2 \\ \frac{2}{\sqrt{3}} & 2 & 0 & -2 \\ \frac{2}{\sqrt{3}} & -2 & -2 & 4 \end{pmatrix}$$

The second order condition in this case looks at LPM_3 and LPM_4. We can factor some positive constants out before finding the LPM's.

$$\begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & -1 \\ 1 & 1 & 0 & -1 \\ 1 & -1 & -1 & 2 \end{pmatrix}$$

Use row operations to simplify a bit.

$$\begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & -1 \\ 0 & 0 & -2 & -1 \\ 0 & 0 & -1 & 4 \end{pmatrix}$$

Hence LPM_3 is 2 and LPM_4 is 9, both positive. Local maximum and local minimum both require LPM_4 to be negative (match the sign of $(-1)^1$ or $(-1)^3$). Hence the point is a saddle. □

(c)

$$\det(A - xI) = \det\begin{pmatrix} -x & 1 & -1 \\ 1 & -x & -1 \\ -1 & -1 & 2-x \end{pmatrix}$$

$$= 2x^2 - x^3 + 1 + 1 + x + x - (2-x) = -x^3 + 2x^2 + 3x = -x(x+1)(x-3)$$

The eigenvalues are $-1, 0, 3$. The corresponding eigenvectors:

$$\begin{pmatrix} 1 & 1 & -1 \\ 1 & 1 & -1 \\ -1 & -1 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow x = t, y = -t, z = 0$$

$$\begin{pmatrix} 0 & 1 & -1 \\ 1 & 0 & -1 \\ -1 & -1 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow x = t, y = t, z = t$$

$$\begin{pmatrix} -3 & 1 & -1 \\ 1 & -3 & -1 \\ -1 & -1 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow x = t, y = t, z = -2t$$

The unit eigenvectors are

$$\begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \\ 0 \end{pmatrix}, \begin{pmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{pmatrix}, \begin{pmatrix} \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \\ \frac{-2}{\sqrt{6}} \end{pmatrix}$$

□

(d)

Because the constraint set is closed and bounded, we can find the maximum value by simply plugging in the 3 solutions and compare. The maximum value of f occurs at $x = \frac{1}{\sqrt{6}}, y = \frac{1}{\sqrt{6}}, z = \frac{-2}{\sqrt{6}}$ and the value is 3. □

(e)

The new Lagrangian function is

$$L(x, y, z, \mu) = 0.03x^2 + 2z^2 + 2xy - 2xz - 2yz - \mu(x^2 + y^2 + z^2 - 1.02).$$

We can use two variables or one.

One variable:

$$L(x, y, z, \mu; a) = 3ax^2 + 2z^2 + 2xy - 2xz - 2yz - \mu(x^2 + y^2 + z^2 - 1 - 2a).$$

Linear approximation at $a = 0.01$ is $3 + (0.01)\left(\frac{\partial L}{\partial a}\right)$ at the point $x = \frac{1}{\sqrt{6}}, y = \frac{1}{\sqrt{6}}, z = \frac{-2}{\sqrt{6}}, \mu = 3$.

$$3 + (0.01)(0.5 + 6) = 3.065.$$

Two variables:

$$L(x, y, z, \mu; a_1, a_2) = a_1x^2 + 2z^2 + 2xy - 2xz - 2yz - \mu(x^2 + y^2 + z^2 - 1 - a_2).$$

At the point $L_{a_1} = \frac{1}{6}$ and $L_{a_2} = 3$. Hence

$$3 + (0.03)\frac{1}{6} + (0.02)3 = 3.065.$$

□

Grading:

- (a) (2%) FOC (students do not need to list all of them, but must explain which one(s) are used), (1%) μ-value.

- (b) (4%) Bordered Hessian. (3%) Classify.
- The answer in (b) depends on the μ-value from (a). Students can get full credit as long as (b) is done correctly with the value from (a). For non-zero μ-values, the student must show work so the grader can see the conclusion of the SOC clearly.
- (c) (3%) for finding eigenvalues. (2%) each for the eigenvectors.
- (d) Student should check NDCQ here, but we won't deduct points if they forget. (2%) for maximum value, all or nothing.
- The answer in (d) depends on the answer in (c). Students can get full credit in (d) even if (c) is incorrect.
- (e) (2%) for writing the new Lagrangian. (3%) for using Envelope Theorem correctly.
- The answer in (e) depends on the answer in (d). Students can get full credit in (e) even if they just used envelope theorem on the given point (which we know is a saddle, not max).
- In general, (-1%) for each minor mistake and (-2%) for each concept mistake. Subtract until there are no points left in the problem.