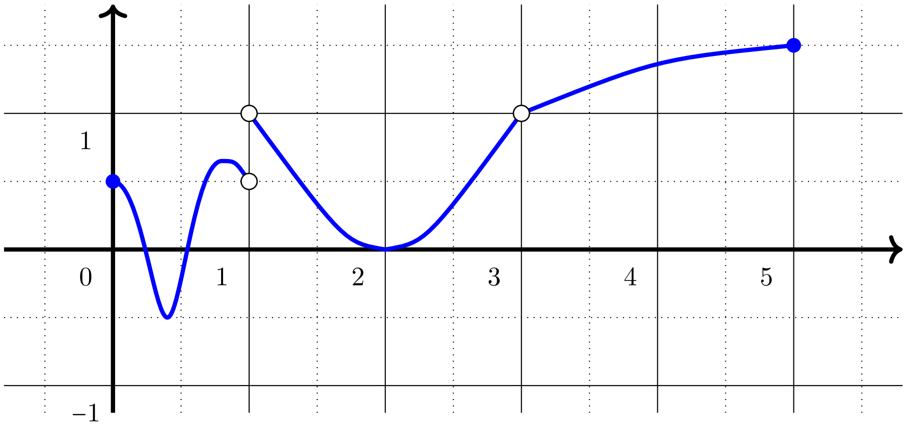
1111模組17-19班 微積分1 期考解答和評分標準

1. (14%) 底下是 $y = f(x)$ 函數圖形。回答底下的填空題與複選題。Use the following graph $y = f(x)$ to answer the following questions.



- $\lim_{x\to 1^+} f(x) =$ __________
- $\lim_{x\to 3^+} f(x) =$ __________
- $\lim_{x\to 1^-} f(x) =$ __________
- $\lim_{x\to 3^-} f(x) =$ __________
- $\lim_{x\to 1} f(x) =$ __________
- $\lim_{x\to 3} f(x) =$ __________

• 函數在用哪個區間當成定義域時會有反函數？（複選） On which interval can we define an inverse function? (Circle all correct choices)

(a) $0 < x < 1$ (b) $1 < x < 2$ (c) $1 < x < 3$ (d) $3 < x < 5$ (e) $2 < x < 5$

Solution: The answers will give from left to right. 2 pts for each question. 1 pt for the last one if he/she misses one answer.

- $\lim_{x\to 1^+} f(x) = \underline{1}$
- $\lim_{x\to 3^+} f(x) = \underline{1}$
- $\lim_{x\to 1^-} f(x) = \underline{\frac{1}{2}}$
- $\lim_{x\to 3^-} f(x) = \underline{1}$
- $\lim_{x\to 1} f(x) =$ not exists
- $\lim_{x\to 3} f(x) = \underline{1}$

(b), (d), (e)

2. (14%) 求極限。(使用 l'Hôpital 法則只能得到部分分數) Evaluate the limit. (Use of L'Hospital's Rule will result in partial credit only)

(a) $\lim_{x\to 3} \frac{x^2-9}{x^2+x-12}$

(b) $\lim_{x\to\infty} x\left(\sqrt{x^2+1}-x\right)$

Solution: (a)

$$\lim_{x\to 3} \frac{x^2-9}{x^2+x-12} = \lim_{x\to 3} \frac{(x+3)(x-3)}{(x-3)(x+4)} = \lim_{x\to 3} \frac{x+3}{x+4} = \frac{6}{7}$$

□

(b)

$$\lim_{x\to\infty} x\left(\sqrt{x^2+1}-x\right) = \lim_{x\to\infty} \frac{x}{\sqrt{x^2+1}+x} = \lim_{x\to\infty} \frac{1}{\sqrt{1+\frac{1}{x^2}}+1} = \frac{1}{2}$$

□

Grading:

It is hard to get this problem wrong, so the grading can be harsh. (2%) if they only found limits of all the parts of the function (example: $x^2-9\to 0$ and $x^2+x-12\to 0$). (4%) if they know that they need to factor or rationalize, but did it incorrectly. (6%) for having only 1 or 2 minor errors like mis-copy or error in simplifying.

Only (3%) for correct answers via l'Hospitals Rule. No points for this part if incorrect.

If they copy the problem wrong and their work show other limit techniques, at most (5%). If the limit became too easy, (3%).

3. (24%) 微分。 Differentiation.

(a) 計算導函數 $f(x) = \frac{\sin x}{\cos^2 x} + \tan^{-1}\left(\sqrt{x}\right)$ (答案不用化簡) 。
Find the derivative of $f(x) = \frac{\sin x}{\cos^2 x} + \tan^{-1}\left(\sqrt{x}\right)$. You do not need to simplify.

(b) 使用隱函數微分來求方程式 $x^3 + 8y^3 = 24xy$ 在點 $(6,3)$ 的一階導數。
A curve is given by $x^3 + 8y^3 = 24xy$. Use implicit differentiation to find the value of $\frac{dy}{dx}$ at the point $(6,3)$.

(c) 給定函數 $g(x) = 2^x \cdot (1+x^2)^{1/x}$。利用對數微分計算導數 $g'(1)$。
Let $g(x) = 2^x \cdot (1+x^2)^{1/x}$. Use logarithmic differentiation to find $g'(1)$.

Solution: 8 pts for each questions.

(a) $\frac{\cos^3 x + 2\sin^2 x \cos x}{\cos^4 x} + \frac{1}{2\sqrt{x}(1+x)}$.
(Any sign of applying quotient rule +2. Any sign of applying chain rule +2. Apply quotient rule correctly +2. Apply chain rule correctly +2)

(b) $3x^2 + 24y^2y' = 24y + 24xy'$. $y' = \frac{3x^2 - 24y}{24x - 24y}$. $y' = \frac{3 \cdot 6^2 - 24 * 3}{24(6-3)} = \frac{1}{2}$.
(Any sign of applying implicit differentiation +2 pt. Correctly, apply differentiation +2 pt. No matter he/she applies implicit differentiation correct or not, as long as he/she tries to solve y' gives 1 pt. Correctly solve y' gives another 1 pt. No matter what he/she get in the latest step. Try plug in (6,3) 1 pt. Correctly evaluate 1 pt.)

(c) $\ln y = x\ln 2 + \frac{1}{x}\ln(1+x^2)$. $\frac{1}{y}y' = \ln 2 + \frac{-1}{x^2\ln(1+x^2)} + \frac{2}{1+x^2}$. $y' = 2^x(1+x^2)^{1/x}(\ln 2 + \frac{-1}{x^2\ln(1+x^2)} + \frac{2}{1+x^2})$.
(+2 if he/she takes ln. +2 for any sign of trying implicit differentiation. +2 if he/she applies implicit differentiation correctly. +2 if he/she solves y' by multiplying y)

4. (12%) 給定函數 $f(x) = \ln(1+x)$。 Let $f(x) = \ln(1+x)$.

(a) 使用在 $x = 0$ 的線性逼近估計 $\ln(0.95)$。
Use the linearization of $f(x)$ at $x = 0$ to approximate the value of $\ln(0.95)$.

(b) 判斷你在(a)部分的估計值與實際的函數值比較時，會比較大還是比較小。詳細解釋你的判斷方法。
Is your estimate in (a) an over- or under-estimate? Justify.

Solution:

(a) $\underbrace{f(0) = 0}_{1M}$ and $\underbrace{f'(x) = \frac{1}{x+1}}_{2M} \Rightarrow \underbrace{f'(0) = 1}_{1M}$.

The linearization of $f(x)$ at $x = 0$ is

$$\underbrace{L(x) = f(0) + f'(0)(x-0)}_{1M} \implies \underbrace{L(x) = x}_{1M}.$$

Hence,

$$\ln(0.95) = \underbrace{f(-0.05)}_{1M} \approx \underbrace{-0.05}_{1M}.$$

(b) $\underbrace{f''(x) = -\frac{1}{(1+x)^2}}_{2M}$.

So $y = f(x)$ is concave downward which means any tangent line lies above the graph of $f(x)$. (1M)

Hence, $\underbrace{L(0.95) > f(0.95)}_{1M}$ (over-estimate).

Marking scheme for 4a
1M - correct value of $f(0)$
2M - correct $f'(x)$
1M - correct $f'(0)$
1M - demonstrating knowledge of linearisation
1M - correct $L(x)$
1M - rewrite $\ln(0.95)$ as $f(-0.05)$
1M - correct estimation
Marking scheme for 4b
2M - correct $f''(x)$
1M - mentioning f is concave downward (accept a picture)
1M - correct conclusion

5. (22%) 給定函數 $f(x) = \dfrac{x^3}{(x+2)^2}$，$x \neq -2$。作圖題。 Consider the function $f(x) = \dfrac{x^3}{(x+2)^2}$ for $x \neq -2$.

(a) 求一階導函數。討論函數之遞增減特徵 (用區間表示) 。
Find $f'(x)$. Write down the interval(s) of increase/decrease of $f(x)$.

(b) 求二階導函數。討論函數之凹性特徵 (用區間表示) 。
Find $f''(x)$. Write down the interval(s) on which $f(x)$ is concave upward/downward.

(c) 討論函數之極值與反曲點 (用座標表示) 。 Write down (if any) the local extrema and inflection points.

(d) 已知函數圖形有一條垂直漸近線與一條斜漸近線。將兩條線的方程式寫出來。
The graph $y = f(x)$ has a vertical asymptote and a slant asymptote. Find them.

(e) 作圖。Sketch the graph $y = f(x)$.

Solution:

(a) (2M) $f'(x) = \dfrac{3x^2(x+2)^2 - x^3 \cdot 2(x+2)}{(x+2)^4} = \dfrac{x^2(x+6)}{(x+2)^3}$.
The critical numbers are -6 and 0.
(1M) $f'(x) > 0$ for $x < -6$ or $x > -2$; $f'(x) < 0$ for $-6 < x < -2$.
(2M) Interval of increase : $(-\infty, -6) \cup (-2, \infty)$, Interval of decrease : $(-6, -2)$

Marking scheme for 5a
2M - correct $f'(x)$ (1M for knowing the quotient rule)
1M - for knowing $f'(x) > 0$ (resp. < 0) implies f is increasing (resp. decreasing)
2M - correct intervals of increase and decrease (partial credits available)

(b) (2M) $f''(x) = \dfrac{(3x^2+12x)(x+2)^3 - (x^3+6x^2)\cdot 3(x+2)^2}{(x+2)^6} = \dfrac{24x}{(x+2)^4}$
(1M) $f''(x) > 0$ when $x > 0$; $f''(x) < 0$ when $x < -2$ or $-2 < x < 0$
(2M) Concave upward : $(0, \infty)$; Concave downward : $(-\infty, -2) \cup (-2, 0)$

Marking scheme for 5b
2M - correct $f''(x)$ (partial credits)
1M - for knowing $f''(x) > 0$ (resp. < 0) implies f is concave upward (resp. downward)
2M - correct intervals of concavity (partial credits available)

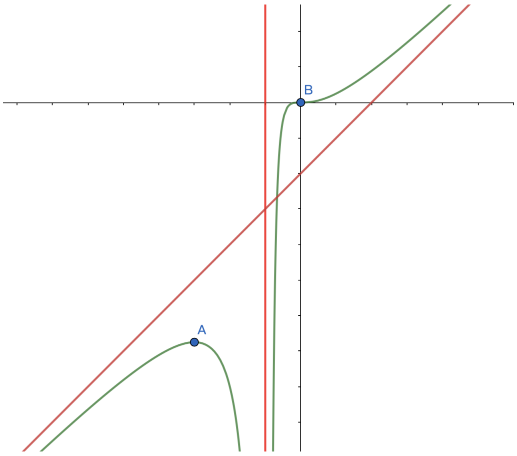
(c) (1M) Local maximum : $(-6, f(-6) = -\dfrac{27}{2})$
Local minimum : NONE
(1M) Inflection point : $(0, f(0) = 0)$

Marking scheme for 5c
0.5M+0.5M - correct local max. (each coordinate)
0.5M+0.5M - correct inflection points (each coordinate)

(d) (1M) $x = -2$ is a vertical asymptote,
(1M) because $\lim_{x\to -2} f(x) = -\infty$
Let $y = ax + b$ be a slant asymptote (towards ∞).
(2M) $a = \lim_{x\to\infty} \dfrac{x^2}{(x+2)^2} = \lim_{x\to\infty} \dfrac{1}{(1+\frac{2}{x})^2} = 1$
(2M) $b = \lim_{x\to\infty} \dfrac{x^3}{(x+2)^2} - x = \lim_{x\to\infty} \dfrac{-4x^2-4}{x^2+4x+4} = \lim_{x\to\infty} \dfrac{-4-\frac{4}{x^2}}{1+\frac{4}{x}+\frac{4}{x^2}} = -4$
(1M) So $y = x - 4$ is a slant asymptote (towards ∞).
The calculation for $x \to -\infty$ is identical.
Hence $y = x - 4$ is the slant asymptote.

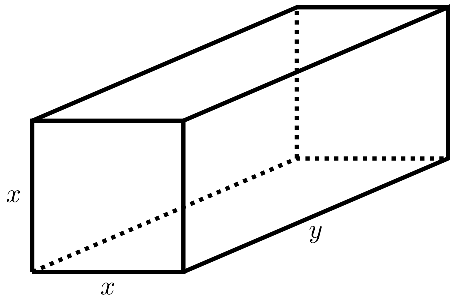
Marking scheme for 5d
1M - correct vertical asymptote
1M - correct verification of the vertical asymptote
2M - computation of a (partial credit available)
2M - computation of b (partial credit available)
1M - correct slant asymptote

(e) Sketch :



Marking scheme for 5e
(0.5M) Asymptotes $x = -2$ and $y = x - 4$
(0.5M) Local max at $(-6, -13.5)$
(0.5M) Inflection point at $(0, 0)$
(0.5M)The shape for $x < -2$
(1M) The shape for $x > 0$

6. (14%) 如圖，一個長方體的盒子有兩面是正方形，邊長分別為 x、x、y。
Consider a rectangular box with a square end as shown in the figure.



若方形面的周長加上長度 $4x + y$ 固定是 3 公尺長，求此設計的盒子體積的最大值。
If the perimeter of the square and the length combined $4x + y$ is equal to 3 meters, what is the largest possible volume of the box?

Solution: Since $y = 3 - 4x$, we can let the volume function be $V(x) = x^2(3 - 4x)$.

The domain is defined by the fact that $x > 0$ and $y > 0$, hence $0 < x < \frac{3}{4}$.

To find the critical number(s), we find the solution to $V'(x) = 0$.

$$V'(x) = 6x - 12x^2 = 6x(1 - 2x)$$

The critical numbers are $x = 0$ and $x = \frac{1}{2}$. Only $x = \frac{1}{2}$ lie in the domain.

We know that $V(0) = 0$ and $V(\frac{3}{4}) = 0$. We can also see that

$$0 < x < \frac{1}{2} \Rightarrow V'(x) > 0$$

$$\frac{1}{2} < x < \frac{3}{4} \Rightarrow V'(x) < 0$$

Therefore the absolute maximum value of the function $V(x)$ over the closed interval $[0, \frac{3}{4}]$ is achieved at $x = \frac{1}{2}$. The maximum volume is equal to

$$V\left(\frac{1}{2}\right) = \frac{1}{2^2}\left(3 - \frac{4}{2}\right) = \frac{1}{4}\text{m}^3$$

□

Grading:

(3%) for setting up a correct function to maximize. There are multiple possible answers. If the function is wrong, they can still get (11%) from later parts.

(1%) for finding the domain that matches common sense. It is okay to use open or closed intervals.

(8%) for the optimization portion: (2%) for derivative, (5%) for critical number(s) and sign chart, (1%) for checking if answer matches domain.

(2%) for presenting final answer (plug in x value into function) with units.

They do not need to quote first or second derivative test (or extreme value theorem), but if their work doesn't make sure that the value they find is a maximum, (-3%).