1. (18%)

(a) Evaluate the following limits.

(i) (6%) $\lim_{x\to 0^-}\left(\sqrt{1+\frac{1}{x^2}}\right)\cdot\sin x$

(ii) (6%) $\lim_{x\to 0}(\cos x)^{\cot^2 x}$

(b) It is known that $g(x)$ is a function such that $g\left(\frac{\pi}{4}\right)=0$ and $g'(x)=\tan x$.

(i) (3%) Use a linearization to estimate the value of $g(0.26\pi)$.

(ii) (3%) Use $g''(x)$ to determine whether the estimate in (b)(i) is an overestimate or underestimate.

Solution:

(a)(i)

$$\lim_{x\to 0^-}\left(\sqrt{1+\frac{1}{x^2}}\right)\cdot\sin x=-\sqrt{\lim_{x\to 0^-}\frac{(1+x^2)\sin^2 x}{x^2}}=-1$$

□

(a)(ii)

$$\lim_{x\to 0}(\cos x)^{\cot^2 x}=e^{\left(\lim_{x\to 0}\cot^2 x\ln(\cos x)\right)}=e^{\left(\lim_{x\to 0}\frac{\ln(\cos x)}{\tan^2 x}\right)}$$

Since

$$\lim_{x\to 0}\ln(\cos x)=0 \quad\text{and}\quad \lim_{x\to 0}\tan^2 x=0$$

Use l'Hospital's Rule on $\frac{0}{0}$ form to get

$$e^{\left(\lim_{x\to 0}\frac{\ln(\cos x)}{\tan^2 x}\right)}=e^{\left(\lim_{x\to 0}\frac{-\tan x}{2\tan x\sec^2 x}\right)}=e^{-1/2}$$

□

(b)(i)

$$L(x)=g\left(\frac{\pi}{4}\right)+g'\left(\frac{\pi}{4}\right)\cdot\left(x-\frac{\pi}{4}\right)$$

$$L(0.26\pi)=0+\tan\frac{\pi}{4}\cdot(0.26\pi-0.25\pi)=0.01\pi$$

□

(b)(ii)

We can find $g''(x)=\sec^2 x$. $g''(\frac{\pi}{4})=2$.

Because the second derivative is positive between 0.25π and 0.26π, the function is concave up between 0.25π and 0.26π and hence the linear approximation is an under-estimate. □

Grading:

(a) (i)

There are many methods for this problem but they are all pretty short, so students get 0%, 3%, or 6% for wrong, incomplete, and correct, respectively.

(a) (ii)

There are a lot of steps in this l'Hospital's Rule problem. We use a step-by-step grading scheme: students get 1% for each correct step toward the answer.

(b) (i)

Linearization is 2% and the estimate is 1%. Students can/might get points for understanding the process even if the answer is wrong.

(b) (ii)

Another step-by-step situation where students must evaluate second derivative and explain what concave up means. 1% for each step toward the answer.

2. (15%) Compute the following derivatives of implicit functions.

(a) (7%) Given : $y^x + e^{x^2} = 2e$. Find $\dfrac{dy}{dx}$ at $(x, y) = (1, e)$.

(b) (8%) Given : $\sin(xy) = \sin x + \sin y$. Find $\dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2}$ at $(x, y) = (0, 0)$.

Solution:

(a) By implicit differentiation, we get

$$\frac{d}{dx} e^{x \ln y} + e^{x^2} 2x = 0 \text{ (2 points)}.$$

Therefore,

$$y^x (\ln y + \frac{x}{y} y') + e^{x^2} 2x = 0 \text{ (3 points)}.$$

Plug in $(x, y) = (1, e)$ to get $e(\ln e + \frac{1}{e} y') + 2e = 0$. So, $y'|_{(1,e)} = -3e$ (2 points).

(b) By implicit differentiation, we get

$$\cos(xy)\big(y + xy'\big) = \cos x + y' \cos y \text{ (2 points)}. \tag{1}$$

Using implicit differentiation again, we get

$$-\sin(xy)\big(y + xy'\big)^2 + \cos(xy)\big(2y' + xy''\big) = -\sin x - (y')^2 \sin y + y'' \cos y \text{ (2 points)}. \tag{2}$$

Plug in $(x, y) = (0, 0)$ in formula (1) to get $y'|_{(0,0)} = -1$ (2 points). Plug in $(x, y) = (0, 0)$ in formula (2) to get $y''|_{(0,0)} = 2y'|_{(0,0)} = -2$ (2 points).

3. (10%) Suppose $f : \mathbb{R} \to \mathbb{R}$ is a function that satisfies

$$f(x+y) = e^{-2y} f(x) + e^{-2x} f(y) \text{ for all real numbers } x, y.$$

(a) (2%) Find $f(0)$.

(b) Suppose in addition that $\lim_{x \to 0} \frac{f(x)}{x} = 3$.

(i) (3%) Find $f'(0)$.

(ii) (5%) If $f(a) = b$, find $f'(a)$ in terms of a and b.

Solution:

(a) Put $\underbrace{x = y = 0}_{1\%}$. Then we have $f(0) = f(0) + f(0) \Rightarrow \underbrace{f(0) = 0}_{1\%}$.

Marking Scheme for Q3a.
1% for putting $x = y = 0$
1% for the answer

(b) (i) $f'(0) = \underbrace{\lim_{h \to 0} \frac{f(h) - f(0)}{h}}_{2\%} = \lim_{h \to 0} \frac{f(h)}{h} = \underbrace{3}_{1\%}$

Marking Scheme for Q3b(i).
2% for the definition of $f'(0)$ as a limit
1% for the answer

(i) Just writing $f'(0) = 3$ without any explanation will receive 1% only.
(ii) Differentiating the given equation/using L'Hospital's rule are invalid because f is not known to be differentiable.

(ii) $f'(a) = \underbrace{\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}}_{1\%} = \underbrace{\lim_{h \to 0} \frac{e^{-2a} f(h) + e^{-2h} f(a) - f(a)}{h}}_{1\%} = \lim_{h \to 0} e^{-2a} \frac{f(h)}{h} + f(a) \frac{e^{-2h} - 1}{h}$

$= e^{-2a} \cdot 3 + f(a) \underbrace{(-2)}_{2\%} = \underbrace{3e^{-2a} - 2b}_{1\%}$

Marking Scheme for Q3b(ii).
1% for the definition of $f'(a)$ as a limit
1% for applying the given functional equation
2% for computing $\lim_{h \to 0} \frac{e^{-2h} - 1}{h} = -2$ (partial credits available)
1% for the answer

Again, differentiating the given equation/ using L'Hospital's rule are not valid unless the candidate has established the differentiability of f (but this is essentially the point of this question).

4. (13%) **Figure 1** shows a *Hydraulic Scissor Lift* for lifting workers to work at various levels and its cross-section. It is known that the two shafts PQ and RS have length 5 m and are hinged at their mid-points. Suppose at time t s, $PR = x$ m and the height of the top platform from the ground is h m. The work platform is lifted upward by moving the shafts such that both PR and SQ decrease at a constant rate of 0.5 m/s.

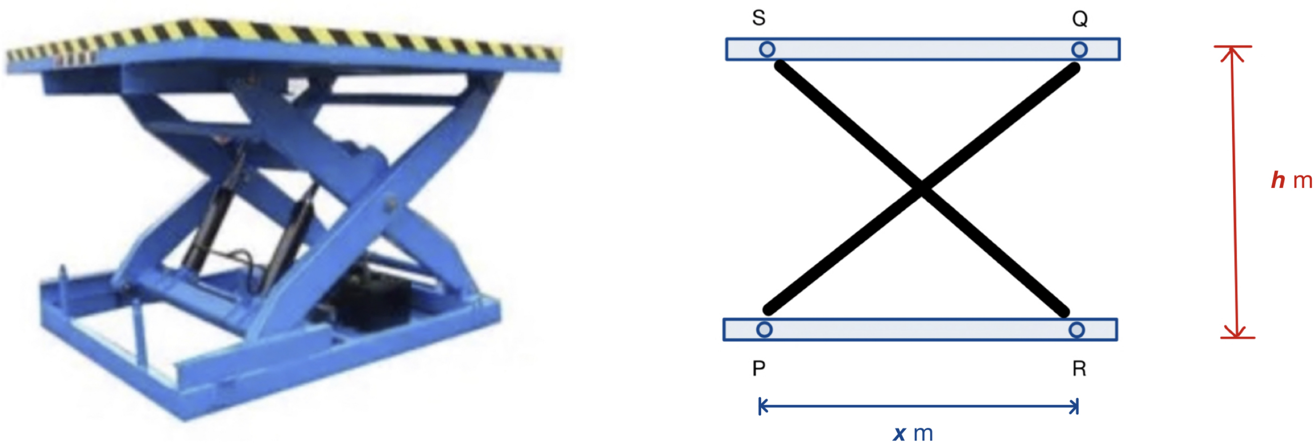


Figure 1. Hydraulic Scissor Lift

(a) (10%) Find $\dfrac{dh}{dt}$ and $\dfrac{d^2h}{dt^2}$ in terms of x.

(b) (3%) According to the government safety regulation, the elevating speed of the platform should not exceed 1 m/s. Find the range of values of x that complies with this regulation.

Solution:

(a) Differentiate $\underbrace{x^2 + h^2 = 25}_{2\%}$ with respect to t yields $\underbrace{2x\dfrac{dx}{dt} + 2h\dfrac{dh}{dt} = 0}_{2\%}$. Putting $\underbrace{\dfrac{dx}{dt} = -0.5}_{1\%}$ and $h = \sqrt{25 - x^2}$ yields

$$-x + 2\sqrt{25 - x^2} \cdot \frac{dh}{dt} = 0 \implies \frac{dh}{dt} = \underbrace{\frac{x}{2\sqrt{25 - x^2}}}_{1\%}$$

Differentiate this again with respect to t,

$$\begin{aligned} \frac{d^2h}{dt^2} &= \frac{\frac{dx}{dt}\sqrt{25 - x^2} - x \cdot \frac{d}{dx}\sqrt{25 - x^2}}{2(25 - x^2)} && (1\%) \\ &= \frac{\frac{dx}{dt}\sqrt{25 - x^2} - x \cdot \frac{-x}{\sqrt{25 - x^2}} \cdot \frac{dx}{dt}}{2(25 - x^2)} && (2\%) \\ &= -\frac{1}{2}\frac{\sqrt{25 - x^2} - x \cdot \frac{-x}{\sqrt{25 - x^2}}}{2(25 - x^2)} \\ &= -\frac{1}{4} \cdot \frac{25}{(25 - x^2)^{3/2}} && (1\%) \end{aligned}$$

Marking Scheme for Q4(a).
First derivative
2% for relating x and h
2% for correct implicit differentiation (partial credits available)
1% for writing $\dfrac{dx}{dt} = -0.5$
1% for the answer
Second derivative
1% for quotient rule
2% for correct implicit differentiation (partial credits available)
1% for the answer

(b) Note $x \geq 0$ by default. Set $\underbrace{\frac{dh}{dt} \leq 1}_{1\%}$. We have $\frac{x}{2\sqrt{25 - x^2}} \leq 1$. Solving gives $\underbrace{x \leq \sqrt{20}}_{2\%}$. Therefore, when the lift is operating for $0 \leq x \leq \sqrt{20}$, it complies with the government regulation.

Marking Scheme for Q4(b).
1% for setting $\frac{dh}{dt} \leq 1$
2% for the correct range of x.

5. (12%)

(a) (6%) Prove that $|\sin b - \sin a| \le |b - a|$ for all real numbers a and b.

(b) (6%) Use (a) to compute $\lim_{x\to\infty}\left(\sin\sqrt{x+\sqrt{x+\sqrt{x}}} - \sin\sqrt{x+\sqrt{x}}\right)$.

Solution:

(a) If $b = a$, both sides are zero, hence the inequality is valid. The case $a < b$ is equivalent to the case $b > a$, so we may assume $b > a$ without loss of generality. By application of Mean Value Theorem to $\sin x$, we find that

$$\frac{\sin b - \sin a}{b - a} = \cos c$$

for some $c \in (a, b)$. Since $\left|\frac{\sin b - \sin a}{b - a}\right| = |\cos c| \le 1$, hence $|\sin b - \sin a| \le |b - a|$.

- Totally 4% for deriving the fact $\frac{\sin b - \sin a}{b - a} = \cos c$ for some $c \in (a, b)$ with explanation. If one uses MVT in the form $\sin b - \sin a = (b - a)\cos c$, dividing the case $a = b$ may be omitted.
  - 1% for examining the case $b = a$ if one uses the quotient form of MVT.
  - 1% for mentioning "by Mean Value Theorem" or "by MVT"
  - 2% for arriving at the fact $\frac{\sin b - \sin a}{b - a} = \cos c$ for some $c \in (a, b)$.
- Totally 2% for obtaining the inequality $|\sin b - \sin a| \le |b - a|$.
  - 1% for using the fact $|\cos c| \le 1$ or $-1 \le \cos c \le 1$.

(b) For $x > 0$,

$$\begin{aligned}
\left|\sin\sqrt{x+\sqrt{x+\sqrt{x}}} - \sin\sqrt{x+\sqrt{x}}\right| &\le \left|\sqrt{x+\sqrt{x+\sqrt{x}}} - \sqrt{x+\sqrt{x}}\right| \quad \text{(by (a))}\\
&= \frac{\left|\left(\sqrt{x+\sqrt{x+\sqrt{x}}} - \sqrt{x+\sqrt{x}}\right)\left(\sqrt{x+\sqrt{x+\sqrt{x}}} + \sqrt{x+\sqrt{x}}\right)\right|}{\left|\sqrt{x+\sqrt{x+\sqrt{x}}} + \sqrt{x+\sqrt{x}}\right|} \quad \text{(by rationalization)}\\
&= \frac{\left|\sqrt{x+\sqrt{x}} - \sqrt{x}\right|}{\left|\sqrt{x+\sqrt{x+\sqrt{x}}} + \sqrt{x+\sqrt{x}}\right|} \quad \text{(by simplification)}\\
&= \frac{\left|\left(\sqrt{x+\sqrt{x}} - \sqrt{x}\right)\left(\sqrt{x+\sqrt{x}} + \sqrt{x}\right)\right|}{\left|\sqrt{x+\sqrt{x+\sqrt{x}}} + \sqrt{x+\sqrt{x}}\right|\left|\sqrt{x+\sqrt{x}} + \sqrt{x}\right|} \quad \text{(by rationalization)}\\
&= \frac{\left|\sqrt{x}\right|}{\left|\sqrt{x+\sqrt{x+\sqrt{x}}} + \sqrt{x+\sqrt{x}}\right|\left|\sqrt{x+\sqrt{x}} + \sqrt{x}\right|} \quad \text{(by simplification)}\\
&\le \frac{\left|\sqrt{x}\right|}{\left|\sqrt{x}\right|\left|2\sqrt{x}\right|} = \frac{1}{4\sqrt{x}}.
\end{aligned}$$

Hence, by Squeeze Theorem,

$$\lim_{x\to\infty}\left(\sin\sqrt{x+\sqrt{x+\sqrt{x}}} - \sin\sqrt{x+\sqrt{x}}\right) = 0 \quad \text{(1pt)}.$$

- 1% for the application of part (a) to derive

$$\left|\sin\sqrt{x+\sqrt{x+\sqrt{x}}} - \sin\sqrt{x+\sqrt{x}}\right| \le \left|\sqrt{x+\sqrt{x+\sqrt{x}}} - \sqrt{x+\sqrt{x}}\right|.$$

- Totally 3% for two-step rationalization with simiplification:

$$\sqrt{x+\sqrt{x+\sqrt{x}}} - \sqrt{x+\sqrt{x}} = \frac{\sqrt{x}}{\left(\sqrt{x+\sqrt{x+\sqrt{x}}} + \sqrt{x+\sqrt{x}}\right)\left(\sqrt{x+\sqrt{x}} + \sqrt{x}\right)}.$$

- 1% for the first rationalization (even for trial).
- 2% for arriving at the simplified form $\sqrt{x+\sqrt{x+\sqrt{x}}}-\sqrt{x+\sqrt{x}}=\dfrac{\sqrt{x+\sqrt{x}}-\sqrt{x}}{\sqrt{x+\sqrt{x+\sqrt{x}}}+\sqrt{x+\sqrt{x}}}$ after the first rationalization.
- 1% for another rationalization and correct simplification

$$\sqrt{x+\sqrt{x+\sqrt{x}}}-\sqrt{x+\sqrt{x}}=\frac{\sqrt{x}}{\left(\sqrt{x+\sqrt{x+\sqrt{x}}}+\sqrt{x+\sqrt{x}}\right)\left(\sqrt{x+\sqrt{x}}+\sqrt{x}\right)}.$$

- Totally 2% for correctly deriving at the conclusion $\lim_{x\to\infty}\left(\sin\sqrt{x+\sqrt{x+\sqrt{x}}}-\sin\sqrt{x+\sqrt{x}}\right)=0$.
  - 1% for the application of Squeeze Theorem.
  - 1% finding its limit of $\dfrac{\sqrt{x}}{\left(\sqrt{x+\sqrt{x+\sqrt{x}}}+\sqrt{x+\sqrt{x}}\right)\left(\sqrt{x+\sqrt{x}}+\sqrt{x}\right)}$ as $x\to\infty$ or for bounding $\dfrac{\sqrt{x}}{\left(\sqrt{x+\sqrt{x+\sqrt{x}}}+\sqrt{x+\sqrt{x}}\right)\left(\sqrt{x+\sqrt{x}}+\sqrt{x}\right)}$ by a quantity that goes to zero as $x\to\infty$.

6. (20%) Consider the function $f(x) = \ln(x^2+1) - 2\ln(x) + 4\tan^{-1}(x) - 4x$ for $x > 0$.

(a) (2%) Find the vertical asymptote of $y = f(x)$.

(b) (6%) Find $\lim_{x\to\infty}(f(x)+4x)$ and hence find the slant asymptote of $y = f(x)$.

(c) (4%) Find $f'(x)$. Write down the interval(s) of increase and interval(s) of decrease of $y = f(x)$.

(d) (4%) Given $f''(x) = \dfrac{-2(x-1)(4x^2+x+1)}{x^2(x^2+1)^2}$. Determine the concavity of $y = f(x)$ and find (if any) point(s) of inflection.

(e) (4%) Sketch the graph of $y = f(x)$. Indicate on your sketch (if any) the local extrema, inflection point(s) and asymptote(s).

Solution:

(a) Since $\lim_{x\to 0^-}\ln x = -\infty$ and $\lim_{x\to 0^-}\ln(x^2+1) + 4\tan^{-1}x - 4x = 0$, we know that $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-}\ln(x^2+1) + 4\tan^{-1}x - 4x - 2\ln x = \infty$. Hence $x = 0$ is a vertical asymptote.
For all $a > 0$, $\lim_{x\to a} f(x) = f(a) \neq \pm\infty$. Hence $x = a$ is not a vertical asymptote for all $a > 0$.
(1 pt for $\lim_{x\to 0^+} f(x) = \infty$. 1 pt for $x = 0$ is the vertical asymptote.)

(b)

$$\begin{aligned}\lim_{x\to\infty} f(x) + 4x &= \lim_{x\to\infty}\ln(x^2+1) - 2\ln x + 4\tan^{-1}x \\ &= \lim_{x\to\infty}\ln\left(\frac{x^2+1}{x^2}\right) + 4\tan^{-1}x \text{ (1 pt for combining } \ln(x^2+1) \text{ and } -2\ln x.) \\ &= \ln\left(\lim_{x\to\infty}\frac{x^2+1}{x^2}\right) + 4\lim_{x\to\infty}\tan^{-1}x \\ &= \ln 1 + 4\times\frac{\pi}{2} = 2\pi \\ &\text{(2 pts for } \lim_{x\to\infty}\ln\left(\frac{x^2+1}{x^2}\right)\text{. 1 pt for } \lim_{x\to\infty}\tan^{-1}x = \frac{\pi}{2}.)\end{aligned}$$

Hence $\lim_{x\to\infty} f(x) + 4x - 2\pi = \lim_{x\to\infty} f(x) - (-4x+2\pi) = 0$ which means that $y = -4x + 2\pi$ is the slant asymptote.
(2 pts for slant asymptotes $y = -4x + 2\pi$)

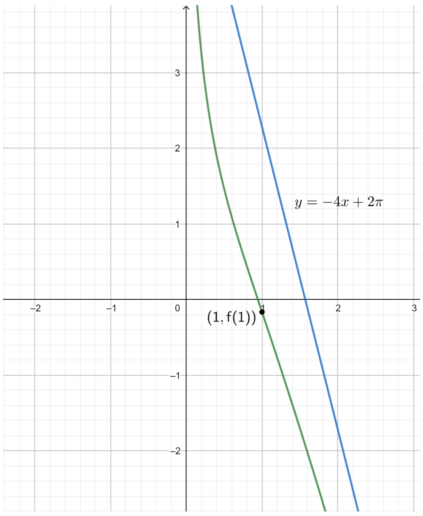
(c)

$$f'(x) = \frac{2x}{x^2+1} - \frac{2}{x} + \frac{4}{x^2+1} - 4 = \frac{2x^2 - 2(x^2+1) + 4x - 4x^3 - 4x}{x(x^2+1)} = \frac{-4x^3-2}{x(x^2+1)} = \frac{-4(x^3+\frac{1}{2})}{x(x^2+1)}$$

(1 pt for differentiating $\ln(x^2+1)$ and $\ln(x)$. 1 pt for differentiating $\tan^{-1}x$.)
For $x > 0$, $f'(x) < 0$. (1 pt for determining the sign of $f'(x)$.)
Hence $f(x)$ is decreasing on $(0,\infty)$. (1 pt)

(d) $f'' = \dfrac{-2(x-1)(4x^2+x+1)}{x^2(x^2+1)^2}$
Because $4x^2+x+1 > 0$, $x^2 \geq 0$, $(x^2+1)^2 > 0$ for all $x \in \mathbb{R}$, we conclude that $f''(x) > 0$ for $x \in (0,1)$ and $f''(x) < 0$ for $x \in (1,\infty)$.
Hence $y = f(x)$ is concave upward on $(0,1)$ and is concave downward on $(1,\infty)$.
$(1, f(1)) = (1, \ln 2 + \pi - 4)$ is the point of inflection.
(1 pt for determining the correct partition $(0,1)$, $(1,\infty)$.
1 pt for the concavity on $(0,1)$.
1 pt for the concavity on $(1,\infty)$.
0.5 pt for the x-coordinate of the inflection point.
0.5 pt for the y-coordinate of the inflection point.)

(e) 0.5 pt for $\lim_{x\to 0^+} f(x) = \infty$. 0.5 pt for the inflection point. 1 pt for the slant asymptote, 1 pt for the graph on $(0,1)$, decreasing and concave upward. 1 pt for the graph on $(1,\infty)$, decreasing and concave downward.



$y = -4x + 2\pi$
$(1, f(1))$

7. (12%) **Figure 2** below shows the intersection of two roads with the same width of 2 m. We want to construct 'Y-shaped' pavements (where $\overline{OB} = \overline{OC}$) as dotted lines shown below. Let $\angle BOC = 2\theta$ with $\frac{\pi}{6} \leq \theta \leq \frac{\pi}{2}$.

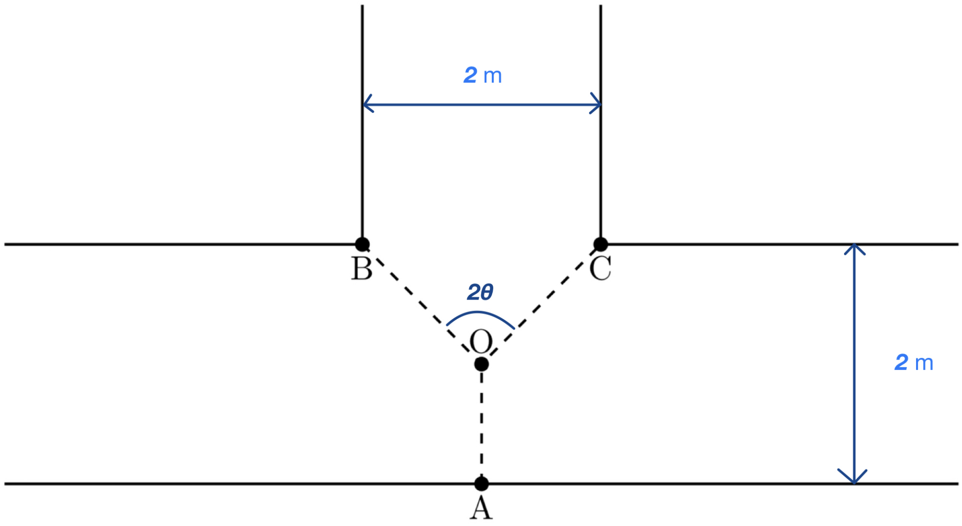


Figure 2.

What is the value of θ that minimizes the total length of pavements, that is, to minimize $\overline{OA} + \overline{OB} + \overline{OC}$?

Solution:

(2 points for expressing the quantities in terms of the notations) Note that $\overline{OB} = \overline{OC} = \csc\theta$ and $\overline{OA} = 2 - \cot\theta$. Then the total length of pavements is

$$f(\theta) = 2\csc\theta + (2 - \cot\theta).$$

(1 point) Taking derivative, we have

$$\text{(4 points*)}\ f'(\theta) = -2\csc\theta\cot\theta + \csc^2\theta = -\csc^2\theta(2\cos\theta - 1).$$

*2 points for derivative of csc and 2 points for derivative of cot. If student tried using quotient rule to find their derivatives, at least 1 point.

(1 point) Thus the critical number of $f(\theta)$ is when $\cos\theta = \frac{1}{2}$, that is $\theta = \frac{\pi}{3}$.

Check $f(\pi/3)$ is the absolute minimum: (4 points)

Method 1. (1 point for mentioning or trying to compute the three values, 2 points for computing correctly, 1 point for comparing with these values with some reason.)

It suffices to compute the following values (of the critical numbers and the endpoints of the interval):

1. $f(\pi/6) = 2 \times 2 + 2 - \sqrt{3} = 6 - \sqrt{3} > 6 - \sqrt{4} = 4.$
2. $f(\pi/3) = 2 \times \frac{2}{\sqrt{3}} + 2 - \frac{1}{\sqrt{3}} = 2 + \sqrt{3} < 2 + \sqrt{4} = 4.$
3. $f(\pi/2) = 2 \times 1 + 2 - 0 = 4$

Since $f(\pi/3)$ is the smallest among them, it is the absolute minimum in the interval $[\pi/6, \pi/2]$.

Method 2. (1 point for trying using monotonicity test**, 1 point for correct sign in each subinterval, 1 point for correctly using monotonicity test, 1 point for correct conclusion, not necessarily correct answer.)

Note that $f'(\theta)$ changes from negative to positive. By the monotonicity test, f is decreasing in the interval $[\pi/6, \pi/3]$ and increasing in the interval $[\pi/3, \pi/2]$. Thus, $f(\pi/3)$ is the absolute minimum in the interval $[\pi/6, \pi/2]$.

**If student only tried to use the first derivative test, then 2 points at most.