1102 Calculus 4 (Class 06-12) Final Exam Solution

June 4, 2022

There are SEVEN questions in this examination.

1. Consider the line integral $I = \oint_C (2xy^2 - x^2y + y^3)\ \mathrm{d}x + (2x^2y - x^3 + 4x)\ \mathrm{d}y$.
   (a) (8%) Use Green's Theorem to evaluate I if C is the closed triangular path with vertices $(0,0)$, $(2,2)$, $(-2,2)$, oriented counterclockwisely.
   (b) (4%) Find the positively oriented simple closed curve C that maximizes I. (You don't have to evaluate I)

Solution:

(a) Since C is a piecewise-smooth simple closed curve that is oriented positively and the vector field is smooth over $\mathbb{R}^2$, we can use Green's Theorem for the region inside C.

$$\int_C \left(2xy^2 - x^2y + y^3\right)\ dx + \left(2x^2y - x^3 + 4x\right)\ dy = \iint_D \left(\frac{\partial}{\partial x}\left(2x^2y - x^3 + 4x\right) - \frac{\partial}{\partial y}\left(2xy^2 - x^2y + y^3\right)\right)\ dA$$

$$= \int_0^2 \int_{-y}^{y} (4 - 2x^2 - 3y^2)\ dx\ dy = \int_0^2 \left(8y - \frac{4y^3}{3} - 6y^3\right)\ dy = 16 - \frac{16}{3} - 24 = -\frac{40}{3}$$

(b) By Green's Theorem, we can maximize I by picking the region where the double integral

$$\iint_D (4 - 2x^2 - 3y^2)\ dA$$

is maximized. Therefore the region D that would maximize I would be $2x^2 + 3y^2 \le 4$, the region where the integrand is positive. The positively oriented simple closed curve C is the boundary of D with a counterclockwise orientation:

$$C:\ \mathbf{r}(t) = \left\langle \sqrt{2}\cos t, \frac{2}{\sqrt{3}}\sin t \right\rangle.$$

Grading:

(a) The points are split into "A clear understanding of Green's Theorem" (3%), "Correctly setting up the iterated integral for the given region" (2%), and "Ability to compute partial derivatives and integration" (3%).

"A clear understanding of Green's Theorem" (3%):

This is shown by having an equation that relates I with a double integral with the correct integrand which can be implied by something as simple as

$$I = \int_0^2 \int_{-y}^{y} (4 - 2x^2 - 3y^2)\ dx\ dy.$$

However, if students fail to imply the equality, write down incorrect integration region or incorrect integrand without any further signs of understanding, then take off (2%) for each mistake (up to (3%)).

Note: Students don't need to check the conditions of the Green's Theorem since it was stated in the problem, but we do encourage students to say it. Don't take points off if students didn't check simple, closed, orientation, or continuously differentiable.

"Correctly setting up the iterated integral for the given region" (2%):

Unless there is a clear minor mistake (-1%), this portion is all or nothing.

"Ability to compute partial derivatives and integration" (3%):

Any mistakes regarding miscopy, sign error, partial derivative, integration, simplify would be in this category. If students used an incorrect Green's Theorem equation or an incorrect double integral, those belong to earlier portions and we should only check if they got the correct computations based on their mistakes. Minor mistakes are (-1%) each and major mistakes are (-2%) each.

Note: Make sure to take points off if students magically got the right computation after having incorrect formula/double integral.

(b) The points are split into "Finding the curve" (2%) and "Justify why it gives a maximum" (2%).

"Finding the curve" (2%):

The answer to this should include the curve as a set (1%) and an orientation (1%). Students do not need to provide a parametrization. We accept describing with words or using a sketch as the answer.

"Justify why it gives a maximum" (2%):

Although we want the students to state Green's Theorem and observe the positivity of the integrand, we accept answers that refer to (a) or the double integral $\int_0^2 \int_{-y}^{y} (4 - 2x^2 - 3y^2)\, dx\, dy$. However, students cannot get this (2%) unless they found $D : 2x^2 + 3y^2 \leq 4$. Partial credit on this part can only happen if they fail to do part (a) or got the wrong integrand, up to the TA in those cases.

Grading TLDR:

(a) (3% + 2% + 3%)

(1) Check if they wrote the conclusion of Green's Theorem. If anything looks wrong (-2%) each.

(2) Check if double integral is set up correctly. (-2%) if not.

(3) Check if computations look correct. (-1%) each or (-2%) if it is something really wrong.

(b) (2% + 2%)

(1) Check if their answer has the curve with the correct orientation, (1%) each.

(2) Check if they explained a little bit.

2. Consider the vector field

$$\mathbf{F}_d = (2x\sin(\pi y) - e^z)\mathbf{i} + (\pi x^2\cos(\pi y) - d\,e^z)\mathbf{j} - xe^z\,\mathbf{k},$$

where d in the **j**-component is a parameter.

(a) (6%) Determine the value of d such that $\mathbf{F}_d$ is conservative in $\mathbb{R}^3$. Find a scalar potential function in this case.

(b) (6%) Evaluate $\int_C \mathbf{F}_4 \cdot d\mathbf{r}$ along the curve C parametrized by $\mathbf{r}(t) = \langle t^3, t^2, t^2\rangle$ with $0 \le t \le 1$.

Solution:

Guideline for Prob2: The following crucial steps must be shown as clear as possible.

(a, 6%): To find $f(x,y,z)$ such that $\nabla f = \vec{F}_d$, one must show that

(∗, 1.5%): $f_x = 2x\sin(\pi y) - e^z \quad \Rightarrow f = x^2\sin(\pi y) - x\,e^z + u(y,z)$

(∗, 1.5%): $f_y = \pi x^2\cos(\pi y) - de^z \quad \Rightarrow f = x^2\sin(\pi y) - d\,y\,e^z + v(x,z)$

(∗, 1.5%): $f_z = -xe^z \quad \Rightarrow f = -xe^z + w(x,y)$.

(∗, 1.5%): Up to a constant, one must have $w = x^2\sin(\pi y),\ v = -xe^z$
$\Rightarrow d = 0$, and $f(x,y,z) = x^2\sin(\pi y) - x\,e^z + \text{constant}$.

(b, 6%): To work out $\int_C \vec{F}_4 \cdot d\vec{r}$, the following steps should be clear

(∗, 1.5%): $\vec{F}_4 = \vec{F}_0 - 4e^z\,\vec{j} = \nabla f - 4e^z\,\vec{j}, \quad d\vec{r} = 3t^2\,\vec{i} + 2t\,\vec{j} + 2t\,\vec{k}$

(∗, 1.5%): $\int_C \vec{F}_4 \cdot d\vec{r} = \int_C \nabla f \cdot d\vec{r} - \int_C 4e^z\,\vec{j}\cdot d\vec{r} = f(\vec{r}(1)) - f(\vec{r}(0)) - \int_0^1 8te^{t^2}\,dt$

(∗, 1.5%): $= f(1,1,1) - f(0,0,0) - 4\int_0^1 e^{t^2}(t^2)'dt$

(∗, 1.5%): $= \sin(\pi) - e - 4\int_0^1 e^t dt = 4 - 5\,e$

3. Consider the 'pringle' surface S (see **Figure 1**) which is part of the graph $z = xy$ inside the cylinder $x^2 + y^2 = 2$. Let C be the boundary of S, counterclockwisely oriented when viewed from above.

   (a) (4%) Set up, but do not evaluate, a definite integral that computes the line integral $\int_C z^2 \, ds$.

   (b) (6%) Find the surface area of S.

   (c) (7%) Use Stokes' Theorem to evaluate

$$\oint_C (x^3 + y)\, dx - 2xy\, dy + (1 + \sin(z^{10}))\, dz.$$

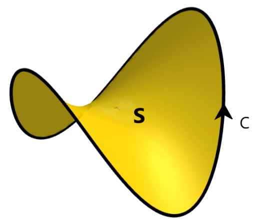

Figure 1. Pringle surface

Solution:

Marking Scheme for Question 3a

- 1% - (1) Paramtrization of C
- 1% - (2) The range of parameters
- 1% - (3) The correct length of $\mathbf{r}'(t)$
- 1% - (3) Setting up the correct integral

Sample Solution to Q3a.

Parametrize C by

$$\mathbf{r}(t) = \langle \sqrt{2}\cos t, \sqrt{2}\sin t, 2\sin t\cos t\rangle = \langle \sqrt{2}\cos t, \sqrt{2}\sin t, \sin(2t)\rangle (1\%), \text{ where } 0 \le t \le 2\pi (1\%)$$

Then $|\mathbf{r}'(t)| = \sqrt{(-\sqrt{2}\sin t)^2 + (\sqrt{2}\cos t)^2 + (2\cos(2t))^2} = \sqrt{2 + 4\cos^2(2t)} (1\%)$ and

$$\int_C z^2 \, ds = \int_0^{2\pi} \sin^2(2t)\sqrt{2 + 4\cos^2(2t)}\, dt (1\%)$$

Marking Scheme for Question 3b

- 1% - (1) Parametrisation of S
- 1% - (2) Correct specification of the ranges of parameters D
- 1% - (3) Correct value of $\|\mathbf{r}_x \times \mathbf{r}_y\|$
  Remark. (2) and (3) will not be awarded if (1) is incorrect.
- 2% - (4) Setting up the correct double integral that evaluates the surface area
  Remark. 1% for students who at least know that surface area equals to $\iint_S 1\, dS$.
- 1% - (5) Correct answer

Sample Solution to Q3(b).

Parametrise S by $\mathbf{r}(x, y) = \underbrace{\langle x, y, xy\rangle}_{(1\%)}$ where $x, y \in \underbrace{D = \{(x, y) : x^2 + y^2 \le 2\}}_{(1\%)}$. Then we have $\|\mathbf{r}_x \times \mathbf{r}_y\| = \sqrt{1 + x^2 + y^2}$

(1%).

$$\iint_S 1\,\mathrm{d}S \overset{\text{def}}{=} \underbrace{\iint_D \sqrt{x^2+y^2+1}\,dA}_{(2\%)} \overset{\text{Polar}}{=} \int_0^{2\pi}\int_0^{\sqrt{2}} r\sqrt{r^2+1}\,dr\,d\theta = \underbrace{\frac{2\pi}{3}\left(3^{\frac{3}{2}}-1\right)}_{(1\%)}.$$

Marking Scheme for Question 3c

- 1% - (1) Statement of Stokes' Theorem
- 1% - (2) Correct computation of curl($\mathbf{F}$)
- 3% - (3) Correct conversion from a flux integral to a double integral.
  Remark. 1% for students who manage to cite
  $$\iint_S \text{curl}\mathbf{F}\cdot dS = \iint_D \text{curl}\mathbf{F}(\mathbf{r}(u,v))\cdot(\mathbf{r}_u\times\mathbf{r}_v)\,dA$$
  But no points if he/she writes 'D' as 'S', and/or '$dA = dudv$' as 'dS' on the RHS
- 2% - (4) Correct evaluation
  Remark. 1% for students who made minor calculation errors/typos.

Sample Solution to Q3(c).

By Stokes' Theorem, $\underbrace{\oint_C \mathbf{F}\cdot d\mathbf{r} = \iint_S \text{curl}(\mathbf{F})\cdot d\mathbf{S}}_{(1\%)}$.

To evaluate the RHS, first we compute that $\underbrace{\text{curl}(\mathbf{F}) = (-2y-1)\mathbf{k}}_{(1\%)}$ and therefore

$$\begin{aligned}\iint_S \text{curl}(\mathbf{F})\cdot d\mathbf{S} = \iint_D \langle 0,0,-2y-1\rangle\cdot\langle -x,-y,1\rangle\,dA &= \underbrace{\iint_D (-2y-1)\,dA}_{(3\%)}\\ &\overset{\text{Polar}}{=} \int_0^{2\pi}\int_0^{\sqrt{2}}(-2r\sin\theta-1)r\,dr\,d\theta\\ &= \underbrace{-2\pi}_{(2\%)}\end{aligned}$$

4. The shell of a spacecraft (see **Figure 2**) is a union of two surfaces :
   - S_1 is part of $x^2 + 4y^2 + 4z^2 = 4$ where $x \le 0$.
   - S_2 is part of $x = 4 - 4y^2 - 4z^2$ where $x \ge 0$.

   Both surfaces are oriented away from the origin.
   Consider the vector field $\mathbf{F}(x, y, z) = 2x\,\mathbf{i} + 2y\,\mathbf{j} - 3z\,\mathbf{k}$.

   (a) (8%) Evaluate, directly, the flux of $\mathbf{F}$ across S_1.

   (b) (8%) Evaluate, directly, the flux of $\mathbf{F}$ across S_2.

   (c) (2%) Use your findings above and the Divergence Theorem to find the volume of the spacecraft.

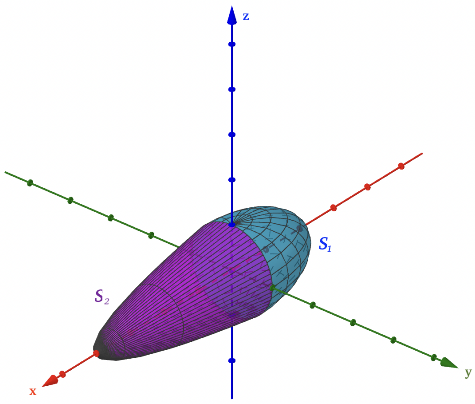

Figure 2. Spacecraft

Solution:

Marking Scheme for Question 4a, 4b

- 2% - (1) Correct parametrisation of $S = S_1$ (or S_2)
- 1% - (2) Correct specification of the ranges of parameters D
- 1% - (3) Correct $\mathbf{r}_u \times \mathbf{r}_v$ up to a sign
  Remark. (2) and (3) will NOT be awarded if (1) is incorrect.
- 2% - (4) Convention of surface integrals into a double integral.
  Remark. 1% for students who manage to cite
  $$\iint_S \mathbf{F} \cdot dS = \iint_D \mathbf{F}(\mathbf{r}(u, v)) \cdot (\mathbf{r}_u \times \mathbf{r}_v)\, dA$$
  But no points if he/she writes 'D' as 'S', and/or '$dA = dudv$' as 'dS' on the RHS
- 2% - (5) Correct evaluation
  Remark. -1% for a sign error

Sample Solution to Q4(a) Ver 1. (Spherical coordinates)

Parametrise S_1 by $\mathbf{r}(\phi, \theta) = \underbrace{\langle 2\cos\theta\sin\phi,\ \sin\theta\sin\phi, \cos\phi\rangle}_{(2\%)}$ where $\underbrace{\frac{\pi}{2} \le \theta \le \frac{3\pi}{2},\ 0 \le \phi \le \pi}_{(1\%)}$. Then

$$\mathbf{r}_\phi = \langle 2\cos\theta\cos\phi,\ \sin\theta\cos\phi,\ -\sin\phi\rangle, \quad \mathbf{r}_\theta = \langle -2\sin\theta\sin\phi,\ \cos\theta\sin\phi, 0\rangle$$
$$\mathbf{r}_\phi \times \mathbf{r}_\theta = \langle \cos\theta\sin^2\phi, 2\sin\theta\sin^2\phi, 2\sin\phi\cos\phi\rangle\ (1\%)$$

$$\begin{aligned}\iint_{S_1} \mathbf{F} \cdot d\mathbf{S} = \int_0^\pi \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \mathbf{F}(\mathbf{r}(\theta, \phi)) \cdot (\mathbf{r}_\phi \times \mathbf{r}_\theta)\, d\theta\, d\phi &= \int_0^\pi \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} -4\sin^3\phi + 6\sin\phi\cos^2\phi\, d\theta\, d\phi (2\%)\\ &= \pi \cdot \int_0^\pi -4\sin\phi + 10\sin\phi\cos^2\phi\, d\phi\\ &= \pi\left[4\cos\phi - \frac{10\cos^3\phi}{3}\right]_0^\pi\\ &= \frac{4\pi}{3} \quad (2\%)\end{aligned}$$

Sample Solution to Q4(a) Ver 2. (Graphs)

Parametrise S_1 by $\mathbf{r}(y,z) = \underbrace{\langle -2\sqrt{1-y^2-z^2}, y, z\rangle}_{2\%}$ where $y,z \in \underbrace{D = \{(y,z) : y^2+z^2 \le 1\}}_{1\%}$. Then

$$\mathbf{r}_z \times \mathbf{r}_y = \left\langle -1, \frac{2y}{\sqrt{1-y^2-z^2}} \frac{2z}{\sqrt{1-y^2-z^2}} \right\rangle \quad (1\%)$$

$$\begin{aligned}
\iint_{S_1} \mathbf{F}\cdot d\mathbf{S} = \iint_D \mathbf{F}(\mathbf{r}(y,z))\cdot(\mathbf{r}_z\times\mathbf{r}_y)\,dA &= \iint_D \left(4\sqrt{1-y^2-z^2} + \frac{4y^2-6z^2}{\sqrt{1-y^2-z^2}}\right) dA \; (2\%) \\
&\overset{\text{Polar}}{=} \int_0^1 \int_0^{2\pi} \left(4r\sqrt{1-r^2} + \frac{4r^3\cos^2\theta - 6r^3\sin^2\theta}{\sqrt{1-r^2}}\right) d\theta\, dr \\
&= \int_0^1 \left(8\pi r\sqrt{1-r^2} - \frac{2\pi r^3}{\sqrt{1-r^2}}\right) dr \\
&= 8\pi \cdot \frac{1}{3} - 2\pi \cdot \frac{2}{3} \\
&= \frac{4\pi}{3} \quad (2\%)
\end{aligned}$$

Sample Solution to Q4(b) Ver 1. (Graphs)

Parametrise S_2 by $\mathbf{r}(y,z) = \underbrace{\langle 4-4y^2-4z^2, y, z\rangle}_{2\%}$ where $y,z \in \underbrace{D = \{(y,z) : y^2+z^2 \le 1\}}_{1\%}$. Then

$$\mathbf{r}_y \times \mathbf{r}_z = \langle 1, 8y, 8z\rangle \quad (1\%)$$

$$\begin{aligned}
\iint_{S_2} \mathbf{F}\cdot d\mathbf{S} = \iint_D \mathbf{F}(\mathbf{r}(y,z))\cdot(\mathbf{r}_t\times\mathbf{r}_z)\,dA &= \iint_D 8 + 8y^2 - 32z^2\, dA \; (2\%) \\
&\overset{\text{Polar}}{=} \int_0^1 \int_0^{2\pi} 8r + 8r^3\cos^2\theta - 32r^3\sin^2\theta\, d\theta\, dr \\
&= \int_0^1 16\pi r - 24\pi r^3 dr \\
&= 2\pi \quad (2\%)
\end{aligned}$$

Sample Solution to Q4(b) Ver 2. (Cylindrical coordinates)

Parametrise S_2 by $\mathbf{r}(r,\theta) = \underbrace{\langle 4-4r^2, r\cos\theta, r\sin\theta\rangle}_{2\%}$ where $\underbrace{0 \le r \le 1,\ 0 \le \theta \le 2\pi}_{1\%}$. Then

$$\mathbf{r}_r \times \mathbf{r}_\theta = \left\langle r, 8r^2\cos\theta, 8r^2\sin\theta\right\rangle \quad (1\%)$$

$$\begin{aligned}
\iint_{S_2} \mathbf{F}\cdot d\mathbf{S} = \int_0^1\int_0^{2\pi} \mathbf{F}(\mathbf{r}(r,\theta))\cdot(\mathbf{r}_r\times\mathbf{r}_\theta)\,d\theta\,dr &= \int_0^1\int_0^{2\pi} \left(8r - 8r^3 + 16r^3\cos^2\theta - 24r^3\sin^2\theta\right) d\theta\, dr (2\%) \\
&= \int_0^1 16\pi r - 24\pi r^3 dr \\
&= 2\pi \quad (2\%)
\end{aligned}$$

Marking Scheme for Question 4c

- 1% - Use Divergence Theorem to deduce that the required volume equals to the sum of the answers from (a) and (b).

  Remark. No marks for students making mistakes in signs here.

- 1% - Correct answer.

Sample Solution to Q4c

Let U be the solid enclosed by S_1 and S_2. By Divergence Theorem, we have

$$\iint_{S_1} \mathbf{F} \cdot \mathrm{d}\mathbf{S} + \iint_{S_2} \mathbf{F} \cdot \mathrm{d}\mathbf{S} = \iiint_U \mathrm{div}(\mathbf{F})\, dV = \iiint_U 1\, dV = \mathrm{Volume}(U) \quad (1\%)$$

Hence, using (a) and (b), we have

$$\mathrm{Volume}(U) = \frac{4\pi}{3} + 2\pi = \frac{10\pi}{3} \quad (1\%)$$

5. (10%) Let $f(x) = \sum_{n=1}^{\infty} \frac{1}{a_1 \cdot a_2 \cdots a_n} \cdot (2x-1)^{3n}$, where $a_n = 8 - \frac{1}{n}$. Find the interval of convergence of $f(x)$.

Solution:

Let $b_n = \frac{1}{a_1 a_2 \cdots a_n}(2x-1)^{3n}$.

$\lim_{n\to\infty} |\frac{b_{n+1}}{b_n}| = \lim_{n\to\infty} \frac{1}{a_{n+1}}|2x-1|^3 = \lim_{n\to\infty} \frac{1}{8 - \frac{1}{n+1}}|2x-1|^3 = \frac{1}{8}|2x-1|^3$.

(2 points for computing $\lim_{n\to\infty} |\frac{b_{n+1}}{b_n}|$.)

By the ratio test, we know that $\sum b_n$ converges absolutely if $\frac{1}{8}|2x-1|^3 < 1$ i.e. $|2x-1| < 2$, and $\sum b_n$ diverges if $\frac{1}{8}|2x-1|^3 > 1$ i.e. $|2x-1| > 2$.

(4 points for applying the ratio test and concluding that the power series behaves differently for $|2x-1| < 2$ and $|2x-1| > 2$)

If $2x-1 = 2$, then $\sum b_n = \sum_{n=1}^{\infty} \frac{8^n}{a_1 \cdots a_n}$

However, $\frac{8^n}{a_1 \cdot a_2 \cdots a_n} = \frac{8}{8-1}\frac{8}{8-\frac{1}{2}}\cdots\frac{8}{8-\frac{1}{n}} > \frac{8}{7}$ for all $n > 1$.

Thus $\lim_{n\to\infty} \frac{8^n}{a_1 \cdot a_2 \cdots a_n} \neq 0$ and we know that $\sum_{n=1}^{\infty} \frac{8^n}{a_1 \cdot a_2 \cdots a_n}$ diverges.

(2 points for discussing the right end point.)

If $2x-1 = -2$, then $\sum b_n = \sum_{n=1}^{\infty} \frac{(-1)^n 8^n}{a_1 \cdots a_n}$.

Again, $\left|\frac{(-1)^n 8^n}{a_1 \cdots a_n}\right| = \frac{8}{8-1}\frac{8}{8-\frac{1}{2}}\cdots\frac{8}{8-\frac{1}{n}} > \frac{8}{7}$ for all $n > 1$.

Thus $\lim_{n\to\infty} \frac{(-1)^n 8^n}{a_1 \cdots a_n} \neq 0$ and we know that $\sum_{n=1}^{\infty} \frac{(-1)^n 8^n}{a_1 \cdots a_n}$ diverges.

(2 points for discussing the left end point.)

(If students try to discuss cases of end points but fail to obtain conclusion, they get 1 point out of 4 points.)

Therefore the series converges if and only if $|2x-1| < 2 \Leftrightarrow -2 < 2x-1 < 2 \Leftrightarrow -\frac{1}{2} < x < \frac{3}{2}$.

The interval of convergence is $(-\frac{1}{2}, \frac{3}{2})$.

6. Let $f(x) = \int_1^{1+x} (t-1)\cdot \ln t \, \mathrm{d}t$.

(a) (5%) Find the Taylor series for $(t-1)\cdot\ln t$ at $t=1$ (Hint : $\ln t = \ln(1+(t-1))$)

(b) (5%) Find the Maclaurin series for $f(x)$. Write down its radius of convergence.

(c) (5%) Approximate the value of $f(0.5)$ up to an error of 10^{-2}. Justify your estimation.

Solution:

(a) First, we have

$$\ln t = \ln(1+(t-1)) = \sum_{n=1}^{\infty}(-1)^{n-1}\frac{(t-1)^n}{n}, (2\%) \text{ for } -1 < t-1 \le 1 (1\%).$$

So the Taylor series for $(t-1)\cdot\ln t$ at $t=1$ is

$$(t-1)\cdot\ln t = \sum_{n=1}^{\infty}(-1)^{n-1}\frac{(t-1)^{n+1}}{n} = \sum_{n=0}^{\infty}\frac{(-1)^n}{n+1}(t-1)^{n+2}, \text{ for } -1 < t-1 \le 1 (2\%).$$

(b) From (a), we know $(t-1)\cdot\ln t$ can represented as a power series in $t-1$ (or $(t-1)\cdot\ln t = \sum_{n=0}^{\infty}\frac{(-1)^n}{n+1}(t-1)^{n+2}$)(1%). So we obtain the Maclaurin series for $f(x)$ as follows.

$$\begin{aligned} f(x) &= \int_1^{1+x}\sum_{n=0}^{\infty}(-1)^n\frac{(t-1)^{n+2}}{n+1}\,dt = \sum_{n=0}^{\infty}\frac{(-1)^n}{n+1}\int_1^{1+x}(t-1)^{n+2}\,dt \ (1\%) \\ &= \sum_{n=0}^{\infty}\frac{(-1)^n}{n+1}\left.\frac{(t-1)^{n+3}}{n+3}\right]_1^{1+x} = \sum_{n=0}^{\infty}\frac{(-1)^n}{(n+1)(n+3)}x^{n+3}. \ (1\%) \end{aligned}$$

(M1) Since the radius of convergence of $(t-1)\cdot\ln t = \sum_{n=0}^{\infty}\frac{(-1)^n}{n+1}(t-1)^{n+2}$ is 1 (1%), we know the radius of convergence of the Maclaurin series for $f(x)$ is 1 (1%).

(M2) Set $a_n = \frac{(-1)^n}{(n+1)(n+3)}x^{n+3}$. Then

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{(n+1)(n+3)}{(n+2)(n+4)}|x| = |x|. \ (1\%)$$

By ratio test, the Maclaurin series for $f(x)$ converges for $|x|<1$ and diverges for $|x|>1$. Therefore, the radius of convergence is 1. (1%)

(c) From (b), since the radius of convergence of the Maclaurin series for $f(x)$ is 1 (or the Maclaurin series for $f(x)$ converges for $|x|<1$), we have

$$f\left(\frac{1}{2}\right) = \sum_{n=0}^{\infty}\frac{(-1)^n}{(n+1)(n+3)}\left(\frac{1}{2}\right)^{n+3}. \ (1\%)$$

Also, we observe that $\sum_{n=0}^{\infty}\frac{(-1)^n}{(n+1)(n+3)}\frac{1}{2^{n+3}}$ is an alternating series and $\{\frac{1}{(n+1)(n+3)2^{n+3}}\}$ is decreasing. So

$$\left|\sum_{n=0}^{k}\frac{(-1)^n}{(n+1)(n+3)}(1/2)^{n+3} - f\left(\frac{1}{2}\right)\right| < \frac{1}{(k+2)(k+4)\cdot 2^{k+4}} \ (2\%)$$

When $k=0$, $\frac{1}{(k+2)(k+4)\cdot 2^{k+4}} = \frac{1}{128} < 10^{-2}$. (1%)

Thus we can use $\frac{1}{24}$ to approximate $f\left(\frac{1}{2}\right)$ and the error is less than 10^{-2}. (1%)

7. (a) (4%) Prove that if $p > 1$ and $0 < q < p$, the series $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n^p - n^q}$ is convergent.

(b) (4%) Determine whether the series $\displaystyle\sum_{n=1}^{\infty} (-1)^{n-1} \cdot a_n$, where $a_n = \begin{cases} \frac{1}{n} & \text{if } n \text{ is odd} \\ \frac{1}{2^n} & \text{if } n \text{ is even} \end{cases}$, is convergent or divergent.

(c) (8%) Find all values of p such that the series $\displaystyle\sum_{n=2}^{\infty} (-1)^n \cdot \frac{(\ln n)^p}{n}$ converges *conditionally*.

Solution:

(a) Use the limit comparison test on $\sum \frac{1}{n^p - n^q}$ and $\sum \frac{1}{n^p}$. Observe that $\frac{n^p}{n^p - n^q} = \frac{1}{1 - n^{q-p}} \to 1$ as $n \to \infty$ because $q < p$ (2 points). The series $\sum_n \frac{1}{n^p}$ converges for $p > 1$, so the given series converges (2 points).

(b) Suppose $\sum (-1)^{n-1} a_n$ is convergent (2 points). Consider

$$b_n = \begin{cases} 0, \text{ if } n \text{ is odd} \\ \dfrac{1}{2^n}, \text{if } n \text{ is even.} \end{cases}$$

Since $\sum b_n \le \sum \frac{1}{2^n}$, $\sum b_n$ converges by the comparison test. Therefore, the following series converges

$$\sum (-1)^{n-1} a_n + \sum b_n = \sum (-1)^{n-1} a_n + b_n.$$

However,

$$(-1)^{n-1} a_n + b_n = \begin{cases} \dfrac{1}{n}, \text{ if } n \text{ is odd} \\ 0, \text{ if } n \text{ is even} \end{cases}$$

which gives rise to the divergent series $\sum \frac{1}{2n-1}$, a contradiction. So $\sum (-1)^{n-1} a_n$ diverges (2 points).

(c) We investigate the alternating series $\displaystyle\sum_{n=2}^{\infty} (-1)^n \frac{(\ln n)^p}{n}$. First, for any given p, $\frac{(\ln n)^p}{n} \to 0$ as $n \to \infty$. Next, consider the function $f(x) = \frac{(\ln x)^p}{x}$. Its first derivative

$$f'(x) = \frac{(\ln x)^{p-1}(p - \ln x)}{x^2}$$

is negative for x large, hence f is decreasing for x large. Therefore, for any given p, $\frac{(\ln n)^p}{n}$ decreases to 0, and so the series $\displaystyle\sum_{n=2}^{\infty} (-1)^n \frac{(\ln n)^p}{n}$ converges (3 points).

We turn to the series $\displaystyle\sum_{n=2}^{\infty} \frac{(\ln n)^p}{n}$. For a given p, $f(x) = \frac{(\ln x)^p}{x}$ is continuous, positive, and decreasing for x large. By the integral test, we can focus on the integral

$$\int_2^{\infty} \frac{(\ln x)^p}{x} dx = \int_{\ln 2}^{\infty} t^p dt$$

where we use the change of variables $\ln x = t$. The last integral diverges if and only if $p \ge -1$. So, $\displaystyle\sum_{n=2}^{\infty} \frac{(\ln n)^p}{n}$ diverges if and only if $p \ge -1$ (3 points).

All together, $\displaystyle\sum_{n=2}^{\infty} (-1)^n \frac{(\ln n)^p}{n}$ converges conditionally if and only if $p \ge -1$ (2 points).