1102 Calculus 4 (Class 01-05) Final Exam Solution

June 4, 2022

There are SEVEN questions in this examination.

1. Determine whether or not the vector field $\mathbf{F}$ is conservative. If it is not, give your reason, if it is, find a function f such that $\mathbf{F} = \nabla f$.

   (a) (4%) $\mathbf{F}(x, y, z) = \left\langle -x + yz, xz + y^2, xy + \sin z \right\rangle$

   (b) (4%) $\mathbf{F}(x, y, z) = \left\langle -z + 2xy, 2yz + x^2, xy^2 + z \right\rangle$

Solution:

(a) Since $\nabla \times \mathbf{F} = \mathbf{0}$ on $\mathbb{R}^3$, the vector field is conservative (**1 pt**). To solve the equation $\nabla f = \mathbf{F}$, integrate the first component of $\mathbf{F}$ to get

$$f(x, y, z) = -\frac{1}{2}x^2 + xyz + g(y, z). \qquad \textbf{(1 pt)}$$

Differentiating the above equation with respect to y gives

$$xz + y^2 = xz + \partial_y g,$$

so $g(y, z) = \frac{1}{3}y^3 + h(z)$ (**1 pt**). After substituting the equation of $g(y, z)$ into the equation of $f(x, y, z)$ and differentiating the resulting equation with respect to z, we obtain

$$xy + \sin z = xy + h'(z),$$

which gives $h(z) = -\cos z + c$ (**1 pt**). Thus,

$$f(x, y, z) = -\frac{1}{2}x^2 + xyz + \frac{1}{3}y^3 - \cos z + c.$$

(b) Since

$$\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z} = 2xy - 2y \neq 0, \qquad \textbf{(2 pts)}$$

or

$$\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x} = -1 - y^2 \neq 0,$$

or

$$\nabla \times \mathbf{F} = \left\langle 2xy - 2y,\ -1 - y^2,\ 0 \right\rangle \neq \mathbf{0},$$

the vector field can not be conservative (**2 pt**s).

2. Let

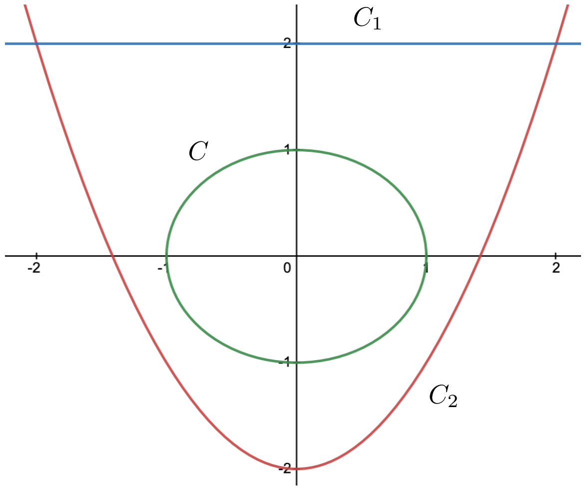
$$\mathbf{F}(x,y) = \langle P(x,y), Q(x,y)\rangle = \left\langle \frac{-y}{x^2+y^2},\ 3x + \sin(y^3) + \frac{x}{x^2+y^2} \right\rangle.$$

(a) (2%) Compute $Q_x - P_y$.

(b) (2%) Compute $\int_{C_1} \mathbf{F}\cdot d\mathbf{r}$ where C_1 is the line segment between $(-2,2)$ and $(2,2)$.

(c) (4%) Compute $\int_{C} \mathbf{F}\cdot d\mathbf{r}$ where C is the counterclockwise-oriented unit circle $x^2+y^2=1$.

(d) (7%) Use Green's Theorem to compute $\int_{C_2} \mathbf{F}\cdot d\mathbf{r}$ where C_2 is the path traveling from $(-2,2)$ to $(2,2)$ along the graph $y = x^2-2$.



Solution:

(a) A direct computation shows that $Q_x - P_y = 3 - 0 = 3$.

(b) 請注意這題題目有一個疏漏：未指定路徑的方向。所以作答時選擇其中一個方向計算就好。以下解答採由$(-2,2)$到$(2,2)$的方向為例。

Method 1: C_1 is the path given by $(s,2)$ with $s\in[-2,2]$, and

$$\int_{C_1}\mathbf{F}\cdot d\mathbf{r} = \int_{-2}^{2}\mathbf{F}(s,2)\cdot\langle 1,0\rangle\,ds = \int_{-2}^{2}\frac{-2}{t^2+4}\,dt = -\arctan(t/2)\Big|_{-2}^{2} = -\frac{\pi}{2}.$$

Method 2: Note that $\mathbf{F} = G + H$ where

$$\mathbf{G}(x,y) = \left\langle \frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2}\right\rangle \quad\text{and}\quad \mathbf{H}(x,y) = \langle 0, 3x+\sin(y^3)\rangle.$$

Since C_1 is horizontal, $\int_{C_1}\mathbf{H}\cdot d\mathbf{r} = 0$. By the angle-counting property of $\mathbf{G}$ we see that

$$\int_{C_1}\mathbf{F}\cdot d\mathbf{r} = \int_{C_1}\mathbf{G}\cdot d\mathbf{r} = \text{ the net change of the polar angle along } C_1 \ = -\frac{\pi}{2}.$$

(c) $C = \partial B_1(0)$ (positively oriented) and is parametrized by $(\cos t, \sin t)$ with $t\in[0,2\pi]$. We have

$$\begin{aligned}\int_C \mathbf{F}\cdot d\mathbf{r} &= \int_0^{2\pi}\mathbf{F}(\cos t,\sin t)\cdot\langle -\sin t,\cos t\rangle\,dt\\ &= \int_0^{2\pi}\left\langle \frac{-\sin t}{\cos^2 t+\sin^2 t}, 3\cos t + \sin(\sin^3 t) + \frac{\cos t}{\cos^2 t+\sin^2 t}\right\rangle\cdot\langle -\sin t,\cos t\rangle\,dt\\ &= \int_0^{2\pi}\left(1+3\cos^2 t\right)dt + \int_0^{2\pi}\sin(\sin^3 t)\cos t\,dt.\end{aligned}$$

We have

$$\begin{aligned}\int_0^{2\pi}\left(1+3\cos^2 t\right)dt &= \int_0^{2\pi}\left(1+3\cdot\frac{1+\cos(2t)}{2}\right)dt\\ &= \int_0^{2\pi}\frac{5}{2}dt + \frac{3}{2}\int_0^{2\pi}\cos(2t)\,dt = 5\pi + 0 = 5\pi\end{aligned}$$

and
$$\int_0^{2\pi} \sin(\sin^3 t)\cos t\,dt = \int_{-\pi}^{\pi} \sin(\sin^3 t)\cos t\,dt = 0 \text{ (the integrand being odd).}$$
In summary, $\int_C \mathbf{F}\cdot d\mathbf{r} = 5\pi$.
(d) "$C_2 - C_1 - C$" bounds the region Ω given by the following system of inequalities:
$$y \geqslant x^2 - 2,\ y \leqslant 2,\quad \text{and}\quad x^2 + y^2 \geqslant 1.$$
By Green's theorem, we have
$$\begin{aligned}\int_{C_2} \mathbf{F}\cdot d\mathbf{r} &= \iint_\Omega (Q_x - P_y)dx\,dy + \int_{C_1} \mathbf{F}\cdot d\mathbf{r} + \int_C \mathbf{F}\cdot d\mathbf{r}\\ &= 3\text{area}(\Omega) + \int_{C_1} \mathbf{F}\cdot d\mathbf{r} + \int_C \mathbf{F}\cdot d\mathbf{r}\\ &= 3\left(\int_{-2}^{2}(2-(x^2-2))dx - \pi\right) - \frac{\pi}{2} + 5\pi = 32 + \frac{3\pi}{2}.\end{aligned}$$

Grading scheme. (a) 與 (b) 皆不給部分分數。(b) 視考生依選擇的路徑方向(或參數化)來決定答案。
(c) 僅在以下情況可得部分分數: 首先將C正確地參數化: $\gamma(t)$ $(t \in [a,b])$。(不一定要跟參考解答所取的相同，例如可取 $(\cos(2t), \sin(2t))$ $(t \in [-\frac{\pi}{2}, \frac{\pi}{2}])$。也可將 C 分段參數化，例如分別考慮上、下半圓弧。)

(1) 正確算出$\mathbf{F}(\gamma(t))\cdot\gamma'(t)$，同時將$\int_a^b \mathbf{F}(\gamma(t))\cdot\gamma'(t)\,dt$正確地分拆成幾個定積分(例如參考答案中的$\int_0^{2\pi}\left(1 + 3\cos^2 t\right)dt$與$\int_0^{2\pi}\sin(\sin^3 t)\cos t\,dt$)，但其中僅有一個定積分算錯或無法求出(例如無法看出$\int_0^{2\pi}\sin(\sin^3 t)\cos t\,dt = 0$)，此時可得3分。

(2) 正確算出$\mathbf{F}(\gamma(t))\cdot\gamma'(t)$，同時將$\int_a^b \mathbf{F}(\gamma(t))\cdot\gamma'(t)\,dt$正確地分拆成幾個定積分(例如參考答案中的$\int_0^{2\pi}\left(1 + 3\cos^2 t\right)dt$與$\int_0^{2\pi}\sin(\sin^3 t)\cos t\,dt$)，但有超過兩個(含)以上的定積分無法正確求得，此時可得2分。

(3) 計算 $\mathbf{F}(\gamma(t))\cdot\gamma'(t)$ 時出錯，最後基於此(錯誤)結果，透過完全正確的過程計算出 $\int_a^b \mathbf{F}(\gamma(t))\cdot\gamma'(t)\,dt$，此時可得2分。

(d) 本質上要將(a)、(b)與(c)所求得之答案套入以下等式來求得 $\int_{C_2} \mathbf{F}\cdot d\mathbf{r}$:
$$(*)\cdots\quad \int_{C_2} \mathbf{F}\cdot d\mathbf{r} = \iint_\Omega (Q_x - P_y)dx\,dy \pm \int_{C_1} \mathbf{F}\cdot d\mathbf{r} + \int_C \mathbf{F}\cdot d\mathbf{r}.$$
若是(a)、(b)或(c)有任何錯誤，此處套用時不重複扣分。給分方式如下:

(1) $(*)$ 中各項均正確列出(不論是寫成抽象的線積分與面積分，或是套入(a)、(b)、(c)所得之具體數字均可)，但等式諸項正負號(與定向有關)有誤。每個項的正負號每錯一個扣2分(最多扣4分，因為四個項的正負號最多只會錯兩個)。

(2) $(*)$ 中各項均有列出(不論是寫成抽象的線積分與面積分，或是套入(a)、(b)、(c)所得之具體數字均可)，且正負號(與定向有關)完全正確，但是套入具體數字時出錯或是有任何代數上的計算錯誤。此時可得6分。

(3) 列出 $(*)$ 時缺少沿 C 的線積分。扣4分。其餘三項正負號有錯錯誤時的扣分原則同 (1)、(2)，(至多)扣2分。

3. Let S be the surface (called a helicoid) shown in the figure.

(a) (2%) The surface is formed by horizontal line segments connecting the z-axis to the helix C:

$$\sigma(t) = (\cos t, \sin t, t), \ 0 \le t \le \pi.$$

Find the x, y, z coordinates of the blue dot if it is distance s from the z-axis at height $z = t$.

(b) (2%) From (a), a parametrization of S is

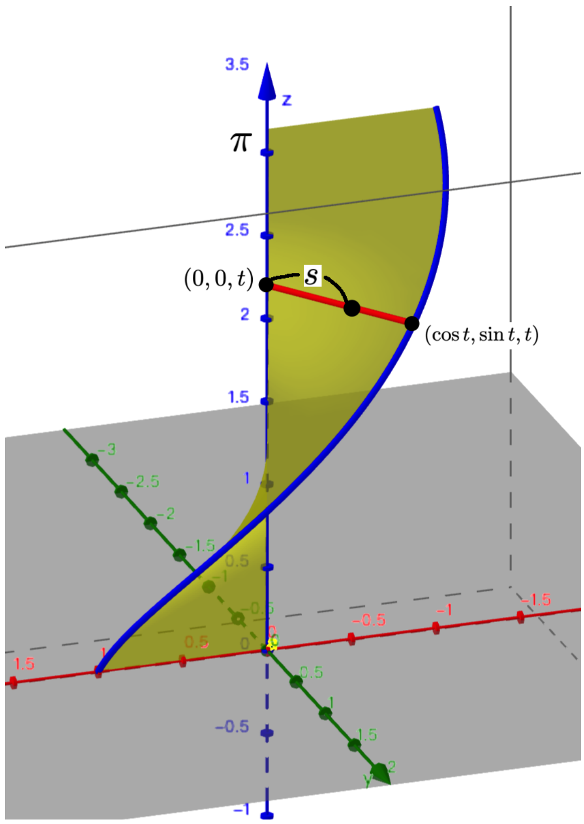
$$\gamma(u,v) = \langle \underline{\qquad\qquad}, \ \underline{\qquad\qquad}, \ \underline{\qquad\qquad} \rangle, \ 0 \le u \le 1, \ 0 \le v \le \pi.$$

(c) (4%) Let

$$\mathbf{F}(x,y,z) = \langle z + e^x, \ 2zy\cos(y^2), \ e^z + \sin(y^2) \rangle$$

Compute $\operatorname{curl} \mathbf{F}$.

(d) (10%) Use Stokes' Theorem to compute the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$.



Solution:

(a) $(s\cos t, s\sin t, t)$.

(b) We may take

$$\gamma(u,v) = (u\cos v, u\sin v, v) \quad (0 \leqslant u \leqslant 1 \ \text{ and } \ 0 \leqslant v \leqslant \pi).$$

(c) A direct computation shows that $\operatorname{curl} F = (0,1,0)$.

(d) We have

$$\gamma_u = \langle \cos v, \sin v, 0 \rangle, \ \gamma_v = \langle -u\sin v, u\cos v, 1 \rangle, \quad \text{and} \quad \gamma_u \times \gamma_v = \langle \sin v, -\cos v, u \rangle.$$

Apply the Stokes theorem: $\iint_S (\operatorname{curl} F) \cdot d\mathbf{S} = \int_{\partial S} \mathbf{F} \cdot d\mathbf{r}$ where S is oriented by the direction of $\gamma_u \times \gamma_v$. We have

$$\int\int_S (\operatorname{curl} \mathbf{F}) \cdot d\mathbf{S} = \int_0^\pi \int_0^1 \langle 0,1,0 \rangle \cdot (\gamma_u \times \gamma_v) \, du \, dv = \int_0^\pi \int_0^1 (-\cos v) du \, dv = 0.$$

On the other hand, $\partial S =$"$C + C_0 - C_1 - C_Z$" where

$$C_0 : (s,0,0)\ (s \in [0,1]),\ C_1 : (-s,0,\pi)\ (s \in [0,1]),\quad \text{and}\quad C_Z : (0,0,t)\ (t \in [0,\pi]).$$

We have

$$\begin{aligned}
&\int_{C_0} \mathbf{F}\cdot d\mathbf{r} = \int_0^1 \mathbf{F}(s,0,0)\cdot\langle 1,0,0\rangle\,ds = \int_0^1 e^s\,ds = e-1,\\
&\int_{C_1} \mathbf{F}\cdot d\mathbf{r} = \int_0^1 \mathbf{F}(-s,0,\pi)\cdot\langle -1,0,0\rangle\,ds = \int_0^1 (-\pi - e^{-s})\,ds = -\pi + \frac{1}{e} - 1,\\
\text{and } &\int_{C_Z} \mathbf{F}\cdot d\mathbf{r} = \int_0^\pi \mathbf{F}(0,0,t)\cdot\langle 0,0,1\rangle\,dt = \int_0^\pi e^t\,dt = e^\pi - 1.
\end{aligned}$$

Therefore,

$$\int_C \mathbf{F}\cdot d\mathbf{r} = -\pi + \frac{1}{e} - 1 + e^\pi - e.$$

Grading scheme. (a) 與 (b) 皆不給部分分數。
(c) 僅在以下情況可得部分分數：如果看得出使用了正確定義

$$\operatorname{curl}\mathbf{F} = \langle R_y - Q_z, P_z - R_x, Q_x - P_y\rangle,$$

此時每個分量算錯扣1分。
(d) 要運用 Stokes 定理，並將邊界積分拆成四段線積分的組合。以參考解答的設定為例，應該要使用以下等式：

$$(*)\cdots\quad \iint_S (\operatorname{curl}\mathbf{F})\cdot d\mathbf{S} = \int_C \mathbf{F}\cdot d\mathbf{r} + \int_{C_0} \mathbf{F}\cdot d\mathbf{r} - \int_{C_1} \mathbf{F}\cdot d\mathbf{r} - \int_{C_Z} \mathbf{F}\cdot d\mathbf{r}.$$

$\operatorname{curl}\mathbf{F}$的 flux 以及$\mathbf{F}$在$C$以外的三段路徑上的線積分時，以及決定正負號時，學生不一定要採用若參考解答中的設定，重點是 Stokes 定理有沒有被正確套用。除了過程全對的情況，其餘情況採扣分制，標準如下：

(1) $\operatorname{curl}\mathbf{F}$的 flux 以及$\mathbf{F}$在$C$以外的三段路徑上的線積分等四個量，依學生所寫的參數化來計算，每個量算錯扣2分。

(2) $(*)$ 中諸項的正負號每錯一個扣2分（最多扣4分，因為有五個項，符號最多只會錯兩個）。

(3) 如果$(*)$中的諸項連同正負號都寫對了，但最後求出$\int_C \mathbf{F}\cdot d\mathbf{r}$時發生計算錯誤，扣1分。

4. Let $f(x,y,z) = x^2+2y^2+3z^2$ and S be the ellipsoid $f(x,y,z) = 1$. The surface integral $\iint_S |\nabla f|\, dS$ can be evaluated with the Divergence Theorem via the steps below.

(a) (2%) Find the outward unit normal vector $\mathbf{n}$ at a point (x,y,z) on S.

(b) (4%) Find a vector field $\mathbf{F}(x,y,z)$ so that

$$\iint_S |\nabla f|\, dS = \iint_S \mathbf{F}\cdot d\mathbf{S}.$$

(c) (9%) Use the Divergence Theorem to compute the surface integral $\iint_S \mathbf{F}\cdot d\mathbf{S}$.

Solution:

(a) The outward unit normal vector on the surface $f = 1$ is given by

$$\mathbf{n} = \frac{\nabla f}{|\nabla f|} = \frac{(x,2y,3z)}{\sqrt{x^2+4y^2+9z^2}}. \qquad \textbf{(2 pts)}$$

(b) Rewrite $|\nabla f|\, dS$ as

$$|\nabla f|\, dS = \nabla f\cdot \frac{\nabla f}{|\nabla f|} dS = \nabla f\cdot d\mathbf{S}. \qquad \textbf{(4 pts)}$$

That is, $\mathbf{F} = \nabla f$.

(c) It follows from (b) and the Divergence Theorem that

$$\iint_S \mathbf{F}\cdot d\mathbf{S} = \iint_{f=1} \nabla f\cdot d\mathbf{S} = \iiint_{f<1} \nabla\cdot\nabla f\, dV, \qquad \textbf{(3 pts)}$$

where integrand of the last term is

$$\nabla\cdot\nabla f = \left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right) f = 2+4+6 = 12. \qquad \textbf{(3 pts)}$$

To evaluating the last triple integral, using the change of variables

$$x = u, \sqrt{2}y = v, \sqrt{3}z = w$$

we get

$$\iiint_{x^2+2y^2+3z^2<1} 12\, dx\, dy\, dz = \iiint_{u^2+v^2+w^2<1} \frac{12}{\sqrt{6}}\, du\, dv\, dw = \frac{12}{\sqrt{6}}\cdot\frac{4}{3}\pi = \frac{8\sqrt{6}}{3}\pi. \qquad \textbf{(3 pts)}$$

5. Determine whether the series is conditionally convergent, absolutely convergent, or divergent.

   (a) (6%) $\sum\limits_{n=2}^{\infty}(-1)^n n!\left(\dfrac{\pi}{n}\right)^n$

   (b) (6%) $\sum\limits_{n=2}^{\infty}\dfrac{(-1)^n}{\left(\sum\limits_{m=1}^{n}\dfrac{1}{m}\right)}$

Solution:

(a) Use Ratio Test: Let $\sum\limits_{n=2}^{\infty}(-1)^n n!\left(\dfrac{\pi}{n}\right)^n = \sum\limits_{n=2}^{\infty}a_n$.

$$a_{n+1} = (-1)^{n+1}(n+1)!\left(\frac{\pi}{n+1}\right)^{n+1} \quad , \quad a_n = (-1)^n n!\left(\frac{\pi}{n}\right)^n$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{(n+1)!}{n!}\frac{\pi^{n+1}}{\pi^n}\frac{n^n}{(n+1)^{n+1}} = \lim_{n\to\infty}\pi\left(\frac{n}{n+1}\right)^n = \pi\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{-n} = \frac{\pi}{e}$$

Since $\pi > e$, the limit $L = \lim\limits_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right| > 1$. Therefore by the Ratio Test we know that the series $\sum\limits_{n=2}^{\infty}a_n$ diverges. $\square$

(b) Let $s_n = \sum\limits_{m=1}^{n}\dfrac{1}{m}$, $a_n = \dfrac{(-1)^n}{s_n}$, and $b_n = |a_n| = \dfrac{1}{s_n}$.

Since $s_{n+1} - s_n = \frac{1}{n+1} > 0$, we know that s_n is increasing and hence b_n is decreasing.

The sequence s_n is the partial sum of the harmonic series, which diverges. So we know that $\lim\limits_{n\to\infty}s_n = \infty$ and $\lim\limits_{n\to\infty}b_n = 0$.

The sequence s_n is positive for any $n \geq 1$, so a_n is alternating and $\sum\limits_{n=2}^{\infty}a_n$ is an alternating series.

With the conditions above, the Alternating Series Test implies that $\sum\limits_{n=2}^{\infty}a_n$ is a convergent series.

Now we consider $\sum\limits_{n=2}^{\infty}b_n$.

Use Comparison Test:

We can see $\frac{1}{m} \leq 1$, so $s_n \leq \sum\limits_{m=1}^{n}1 = n$, which means $b_n \geq \frac{1}{n}$.

Since $\sum\limits_{n=2}^{\infty}\dfrac{1}{n}$ is the harmonic series, it diverges.

From the Comparison Test, we see that $\sum\limits_{n=2}^{\infty}b_n$ is also a divergent series.

Therefore $\sum\limits_{n=2}^{\infty}a_n$ is conditionally convergent.

Note: Students can also compare s_n with $\ln n$. In that case it is okay if the student did not check whether the harmonic series satisfy the Integral Test conditions.

Grading:

No points for "guessing" the right answer. Both series give inconclusive answers to other tests, so any other method can only get at most (2%) (up to TA's decision to decide if their work is really worth partial credits). If students use Stirling's formula for (a), they can get full credit if they stated the formula accurately, otherwise it is (-4%). For (b), if students only checked the alternating or the absolute value series, it is at most (2%).

Any mistake, missing condition, wrong conclusion, or other errors would be (-2%) each until no points are left.

6. Find the interval of convergence of the series.

(a) (8%) $\displaystyle\sum_{n=0}^{\infty} \frac{(x-1)^n}{(-2)^n\sqrt{n^2+1}}$

(b) (8%) $\displaystyle\sum_{n=1}^{\infty} \frac{\tan^{-1}(2^{-n})}{n} x^n$

Solution:

(a) Let $a_n = \dfrac{(x-1)^n}{(-2)^n\sqrt{n^2+1}}$. Use Ratio Test on the power series:

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{\sqrt{n^2+1}}{2\sqrt{n^2+2n+2}}|x-1| = \frac{|x-1|}{2}$$

$$\frac{|x-1|}{2} < 1 \;\Rightarrow\; -1 < x < 3. \text{ Centered at 1. Radius of convergence is 2.}$$

For $x=-1$: Checks $\sqrt{n^2+1}$ is positive. Checks $\displaystyle\lim_{n\to\infty}\frac{n}{\sqrt{n^2+1}} = 1$. Checks $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n}$ diverges.

$\displaystyle\sum_{n=0}^{\infty}\frac{1}{\sqrt{n^2+1}}$ diverges by Limit Comparison Test (or also Integral Test or Comparison Test).

For $x=3$: Checks $\sqrt{n^2+1}$ is positive and increasing. Checks $\displaystyle\lim_{n\to\infty}\frac{1}{\sqrt{n^2+1}} = 0$.

$\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n}{\sqrt{n^2+1}}$ converges by Alternating Series Test.

Therefore the interval of convergence is $(-1,3]$ □

(b) Let $a_n = \dfrac{\tan^{-1}(2^{-n})}{n}x^n$. Use Ratio Test on the power series:

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{\tan^{-1}(2^{-n-1})}{\tan^{-1}(2^{-n})}\frac{n}{n+1}|x|$$

For the portion with the inverse tangent function, we use the L'Hospital's Rule for $\frac{0}{0}$.

$$\lim_{x\to\infty}\frac{\tan^{-1}(2^{-x-1})}{\tan^{-1}(2^{-x})} = \lim_{x\to\infty}\frac{\frac{1}{1+2^{-2x-2}}2^{-x-1}(-\ln 2)}{\frac{1}{1+2^{-2x}}2^{-x}(-\ln 2)} = \frac{1}{2}.$$

Hence $\displaystyle\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \frac{1}{2}$.

$$\frac{|x|}{2} < 1 \;\Rightarrow\; -2 < x < 2. \text{ Centered at 0. Radius of convergence is 2.}$$

For $x=-2$: Checks $\dfrac{2^n\tan^{-1}(2^{-n})}{n}$ is positive and decreasing.

$$\frac{d}{dt}\frac{2^t\tan^{-1}\left(2^{-t}\right)}{t} = \frac{t(\ln 2)2^t\tan^{-1}\left(2^{-t}\right) + \frac{-t(\ln 2)}{1+2^{-2t}} - 2^t\tan^{-1}\left(2^{-t}\right)}{t^2}$$

$$= \frac{\left[2^t(1+2^{-2t})\tan^{-1}(2^{-t})\right]((\ln 2)t - 1) - (\ln 2)t}{t^2(1+2^{-2t})}$$

Use $\displaystyle\tan^{-1}x = \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{2n+1}$ and let $y = 2^{-t}$ to get

$$2^t(1+2^{-2t})\tan^{-1}(2^{-t}) = \frac{1+y^2}{y}\tan^{-1}(y) = (1+y^2)\sum_{n=0}^{\infty}(-1)^n\frac{y^{2n}}{2n+1}$$

$$= \sum_{n=0}^{\infty}(-1)^n\frac{y^{2n}}{2n+1} + \sum_{n=0}^{\infty}(-1)^n\frac{y^{2n+2}}{2n+1} = 1 + \sum_{n=0}^{\infty}(-1)^{n+1}\frac{y^{2n+2}}{2n+3} + \sum_{n=0}^{\infty}(-1)^n\frac{y^{2n+2}}{2n+1}$$

$$= 1 + \sum_{n=0}^{\infty} (-1)^n \left(\frac{1}{2n+1} - \frac{1}{2n+3} \right) y^{2n+2} < 1 + \frac{2y^2}{15}$$

from observing it is an alternating series.

Hence the derivative

$$\frac{d}{dt} \frac{2^t \tan^{-1}\left(2^{-t}\right)}{t} < \frac{\left[1 + \frac{2}{15} 2^{-2t}\right]\left((\ln 2)t - 1\right) - (\ln 2)t}{t^2\left(1 + 2^{-2t}\right)} = \frac{\frac{2(\ln 2)t}{15} 2^{-2t} - 1 - \frac{2}{15} 2^{-2t}}{t^2\left(1 + 2^{-2t}\right)} < 0$$

since $t < 2^{2t}$ for $t \geq 1$.

Checks $\lim_{n\to\infty} 2^n \tan^{-1}\left(2^{-n}\right) = 1$, so $\lim_{n\to\infty} \frac{2^n \tan^{-1}\left(2^{-n}\right)}{n} = 0$.

$(-1)^n \frac{2^n \tan^{-1}\left(2^{-n}\right)}{n}$ converges by Alternating Series Test.

For $x = 2$: Checks $\frac{2^n \tan^{-1}\left(2^{-n}\right)}{n}$ is positive. Checks $\lim_{n\to\infty} 2^n \tan^{-1}\left(2^{-n}\right) = 1$. Checks $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

$\sum_{n=0}^{\infty} \frac{2^n \tan^{-1}\left(2^{-n}\right)}{n}$ diverges by Limit Comparison Test.

Therefore the interval of convergence is $[-2, 2)$

Grading:

The points for both parts are split into "Finding the open interval" (2%), "Check endpoints" (2%) each, and "No mistakes or missing conditions" (2%).

"Finding the open interval" (2%): As long as students use Ratio or Root Test properly, they can get (2%).

"Check endpoints" (2%): Lose (2%) for using the wrong test or have incorrect conclusion.

"No mistakes or missing conditions" (2%): If students show knowledge about the process but make mistakes, then take the (-2%) from here. TA can decide on (-1%) for poor exposition (hard to read, mixing up conditions and conclusions, confusing statements). Similarly for students writing $\sum_{n=0}^{\infty} \frac{1}{n}$ or $\sum a_n$.

Note: Since it is a little complicated to check for decreasing in (b), as long as the student attempts to explain why AST works, they can get the credit.

7. Let $f(x) = \int_0^x \sqrt[3]{8+t^3}\, dt$. (Use the binomial coefficients notation $\binom{k}{n}$ for your answers.)

(a) (10%) Write down the Maclaurin series for $f(x)$ and find its radius of convergence.

Solution:
$f(x) = \int_0^x \sqrt[3]{8+t^3}\, dt = \int_0^x 2(1+\frac{t^3}{8})^{\frac{1}{3}}\, dt$
For $|\frac{t^3}{8}| < 1$ i.e. $|t| < 2$, $(1+\frac{t^3}{8})^{\frac{1}{3}} = \sum_{n=0}^{\infty} \binom{\frac{1}{3}}{n}(\frac{t^3}{8})^n$.
Hence by the term-by-term integration theorem, we know that for $|x| < 2$,

$$f(x) = 2\int_0^x \sum_{n=0}^{\infty} \binom{\frac{1}{3}}{n}(\frac{t^3}{8})^n\, dt = \sum_{n=0}^{\infty} 2\binom{\frac{1}{3}}{n}\frac{1}{3n+1}\frac{1}{2^{3n}}x^{3n+1} = \sum_{n=0}^{\infty} \binom{\frac{1}{3}}{n}\frac{1}{3n+1}\frac{1}{2^{3n-1}}x^{3n+1}.$$

Let $a_n = \binom{\frac{1}{3}}{n}\frac{1}{3n+1}\frac{1}{2^{3n-1}}x^{3n+1}$.
$\because \left|\frac{a_{n+1}}{a_n}\right| = \frac{n-\frac{1}{3}}{n+1}\frac{3n+1}{3n+4}\frac{1}{2^3}|x|^3 \to \frac{|x|^3}{2^3}$ as $n \to \infty$.
$\therefore$ we derive that the radius of convergence of $f(x)$ is 2.
Or because the radius of convergence of the Maclaurin series for $\sqrt[3]{8+t^3}$ is 2, the integration of this Maclaurin series still has radius of convergence 2.
(2 points for writing $\sqrt[3]{8+t^3} = 2(1+\frac{t^3}{8})^{\frac{1}{3}}$.
2 points for the power series representation of $(1+\frac{t^3}{8})^{\frac{1}{3}}$.
3 points for term-by-term integration and correct Maclaurin series for $f(x)$.
3 points for the radius of convergence.)

(b) (2%) Find $f^{(25)}(0)$.

Solution:
We know that $\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n = \sum_{n=0}^{\infty} \binom{\frac{1}{3}}{n}\frac{1}{(3n+1)2^{3n-1}}x^{3n+1}$
Compare the coefficient in front of x^{25} and we obtain $\frac{f^{(25)}(0)}{25!} = \binom{\frac{1}{3}}{8}\frac{1}{25\times 2^{23}} \Rightarrow f^{(25)}(0) = \binom{\frac{1}{3}}{8}\frac{24!}{2^{23}}$

1 point for checking the coefficient of x^{25}.
1 point for correct answer.

(c) (4%) Use a partial sum to estimate $f\left(\frac{2}{3}\right)$ correct to within 10^{-4}.

Solution:
$f(\frac{2}{3}) = \sum_{n=0}^{\infty} \binom{\frac{1}{3}}{n}\frac{1}{(3n+1)2^{3n-1}}(\frac{2}{3})^{3n+1} = 4\sum_{n=0}^{\infty} \binom{\frac{1}{3}}{n}\frac{1}{3n+1}(\frac{1}{3})^{3n+1}$
For $n \geq 1$, the series is an alternating series.
And $\{|\binom{\frac{1}{3}}{n}\frac{1}{3n+1}\frac{1}{3^{3n+1}}|\}$ is decreasing and tends to 0 as n approaches ∞.
Thus by the alternating series estimation, for $k \geq 1$

$$\left|f(\frac{2}{3}) - 4\sum_{n=0}^{k}\binom{\frac{1}{3}}{n}\frac{1}{3n+1}(\frac{1}{3})^{3n+1}\right| \leq \left|4\binom{\frac{1}{3}}{k+1}\frac{1}{3k+4}\frac{1}{3^{3k+4}}\right|$$

Observe that for $k = 1$, $k+1 = 2$,

$$\left|4\binom{\frac{1}{3}}{2}\frac{1}{7}\frac{1}{3^7}\right| = \frac{4}{7\times 3^9} < \frac{1}{3^9} < 10^{-4}.$$

And the partial sum of the first two terms are

$$4\sum_{n=0}^{1}\binom{\frac{1}{3}}{n}\frac{1}{3n+1}\frac{1}{3^{3n+1}} = \frac{4}{3} + \frac{4}{3}\times\frac{1}{4}\times\frac{1}{3^4} = \frac{4}{3} + \frac{1}{3^5}$$

Hence we know that $|f(\frac{2}{3}) - (\frac{4}{3} + \frac{1}{3^5})| < 10^{-4}$.

1 point for observing that the series is an alternating series and satisfies hypothesis of the alternating series estination. 1 point for using the alternating series estimation.

2 points for computing the partial sum and showing that the error is less than 10^{-4}.

Students can use partial sums with more terms, for example, $\frac{4}{3} + \frac{1}{3^5} - \frac{4}{7 \times 3^9}$, but they need to show that the error is less than 10^{-4}.