1. Let $g(x) = \int_4^{2x^2} \frac{t}{1+t^3}\,dt$ and $f(x) = \ln(2+g(x))$.

   (a) (4%) Find $g'(x)$.

   (b) (4%) Find $f'(\sqrt{2})$.

Solution:

(a) By FTC, $g'(x) = \underbrace{\frac{2x^2}{1+8x^6}}_{(2M)} \cdot \underbrace{4x}_{(2M)}$

Marking scheme :

- 2M for the term by replacing t with $2x^2$
- 2M for the term coming from the Chain rule

(b) By chain rule, $f'(x) = \underbrace{\frac{g'(x)}{2+g(x)}}_{(2M)}$ Since $\underbrace{g(\sqrt{2}) = 0}_{(1M)}$, we have $f'(\sqrt{2}) = \frac{g'(\sqrt{2})}{2} = \underbrace{\frac{8\sqrt{2}}{65}}_{(1M)}$ Marking scheme :

- 2M for differentiating $f(x)$ correctly
- 1M for knowing $g(\sqrt{2}) = 0$
- 1M for answer

2. Evaluate the following integrals.

(a) (8%) $\int (1-x^2)^{\frac{3}{2}} \cdot x^3 \,\mathrm{d}x$ (b) (10%) $\int \frac{e^x}{(e^x+1)^2(e^{2x}+1)} \mathrm{d}x$

Solution:

(a) Method1:
Let $x = \sin(\theta)$.
Then $dx = \cos(\theta)d\theta$ and $(1-x^2)^{\frac{3}{2}}x^3 = (1-\sin(\theta))^{\frac{3}{2}}\sin^3(\theta) = \cos^3(\theta)\sin^3(\theta)$.
Thus

$$\int (1-x^2)^{\frac{3}{2}}x^3dx = \int \cos^3(\theta)\sin^3(\theta)\cos(\theta)d\theta$$
$$= \int \cos^4(\theta)\sin^3(\theta)d\theta = \int \cos^4(\theta)\sin^2(\theta)\sin(\theta)d\theta = \int \cos^4(\theta)(1-\cos^2(\theta))\sin(\theta)d\theta$$

Let $u = \cos(\theta)$. Then $du = -\sin(\theta)d\theta$ and $\sin(\theta)d\theta = -du$.
So

$$\int \cos^4(\theta)(1-\cos^2(\theta))\sin(\theta)d\theta$$
$$= \int u^4(1-u^2)(-du) = \int -u^4+u^6du = -\frac{u^5}{5}+\frac{u^7}{7}+C = -\frac{\cos^5(\theta)}{5}+\frac{\cos^7(\theta)}{7}+C$$

Recall that $\sin(\theta) = x$. We have $\cos(\theta) = \sqrt{1-x^2}$. Thus $-\frac{\cos^5(\theta)}{5}+\frac{\cos^7(\theta)}{7} = -\frac{(1-x^2)^{\frac{5}{2}}}{5}+\frac{(1-x^2)^{\frac{7}{2}}}{7}$ and

$$\int (1-x^2)^{\frac{3}{2}}x^3dx = -\frac{(1-x^2)^{\frac{5}{2}}}{5}+\frac{(1-x^2)^{\frac{7}{2}}}{7}+C.$$

1 point for the correct substitution $x = \sin(\theta)$ and $dx = \cos(\theta)d\theta$,
1 point for getting $\int (1-x^2)^{\frac{3}{2}}x^3dx = \int \cos^4(\theta)\sin^3(\theta)d\theta$
1 point for the substitution $u = \cos(\theta)$ and $du = -\sin(\theta)d\theta$,

1 point for getting $\int \cos^4(\theta)(1-\cos^2(\theta))\sin(\theta)d\theta = \int -u^4+u^6du$
1 point for $\int -u^4+u^6du = -\frac{u^5}{5}+\frac{u^7}{7}+C$
1 point for getting $-\frac{\cos^5(\theta)}{5}+\frac{\cos^7(\theta)}{7}+C$
1 point for getting $\cos(\theta) = \sqrt{1-x^2}$
1 point for the final answer $\int (1-x^2)^{\frac{3}{2}}x^3dx = -\frac{(1-x^2)^{\frac{5}{2}}}{5}+\frac{(1-x^2)^{\frac{7}{2}}}{7}+C.$

Method2:
Let $u = 1-x^2$.
Then $du = -2xdx$, $xdx = -\frac{1}{2}du$, $x^2 = 1-u$ and
$(1-x^2)^{\frac{3}{2}}x^3dx = (1-x^2)^{\frac{3}{2}}x^2xdx = \int u^{\frac{3}{2}}(1-u)(-\frac{1}{2})du = \frac{-u^{\frac{3}{2}}+u^{\frac{5}{2}}}{2}du.$

$$\int (1-x^2)^{\frac{3}{2}}x^3dx = \int \frac{-u^{\frac{3}{2}}+u^{\frac{5}{2}}}{2}du$$
$$= -\frac{u^{\frac{5}{2}}}{5}+\frac{u^{\frac{7}{2}}}{7}+C = -\frac{(1-x^2)^{\frac{5}{2}}}{5}+\frac{(1-x^2)^{\frac{7}{2}}}{7}+C.$$

1 point for the correct substitution $u = 1-x^2$
1 point for $du = -2xdx$,
1 point for getting $x^2 = 1-u$
2 point for getting $\int (1-x^2)^{\frac{3}{2}}x^3dx = \int u^{\frac{3}{2}}(1-u)(-\frac{1}{2})du$

2 point for getting $\int \frac{-u^{\frac{3}{2}} + u^{\frac{5}{2}}}{2} du = -\frac{u^{\frac{5}{2}}}{5} + \frac{u^{\frac{7}{2}}}{7} + C$

1 point for the final answer $\int (1-x^2)^{\frac{3}{2}} x^3 dx = -\frac{(1-x^2)^{\frac{5}{2}}}{5} + \frac{(1-x^2)^{\frac{7}{2}}}{7} + C.$

Method3:

Let $u = x^2$ and $dv = (1-x^2)^{\frac{3}{2}} x$. Then $du = 2xdx$, $v = \int (1-x^2)^{\frac{3}{2}} xdx = -\frac{1}{5}(1-x^2)^{\frac{5}{2}}$.

$$
\begin{aligned}
&\int (1-x^2)^{\frac{3}{2}} x^3 dx = \int x^2 (1-x^2)^{\frac{3}{2}} xdx \\
=&x^2(-\frac{1}{5}(1-x^2)^{\frac{5}{2}}) + \frac{1}{5}\int (1-x^2)^{\frac{5}{2}} 2xdx \\
=&-\frac{x^2}{5}(1-x^2)^{\frac{5}{2}} - \frac{1}{5}\cdot\frac{2}{7}\cdot(1-x^2)^{\frac{7}{2}} + C \\
=&-\frac{x^2}{5}(1-x^2)^{\frac{5}{2}} - \frac{2(1-x^2)^{\frac{7}{2}}}{35} + C.
\end{aligned}
$$

Remark: We can simplify the answer

$$
\begin{aligned}
&-\frac{x^2}{5}(1-x^2)^{\frac{5}{2}} - \frac{2(1-x^2)^{\frac{7}{2}}}{35} = (1-x^2)^{\frac{5}{2}}(-\frac{x^2}{5} - \frac{2(1-x^2)}{35}) \\
=& (1-x^2)^{\frac{5}{2}}(-\frac{1}{5} + \frac{1-x^2}{5} - \frac{2(1-x^2)}{35}) \\
=& -\frac{(1-x^2)^{\frac{5}{2}}}{5} + \frac{(1-x^2)^{\frac{7}{2}}}{7}
\end{aligned}
$$

1 point for $u = x^2$, $du = 2xdx$,

1 point for getting $dv = (1-x^2)^{\frac{3}{2}} x$

2 point for getting $v = -\frac{1}{5}(1-x^2)^{\frac{5}{2}}$

2 point for getting $\int (1-x^2)^{\frac{3}{2}} x^3 dx = x^2(-\frac{1}{5}(1-x^2)^{\frac{5}{2}}) + \frac{1}{5}\int (1-x^2)^{\frac{5}{2}} 2xdx$

2 point for the final answer $\int (1-x^2)^{\frac{3}{2}} x^3 dx = -\frac{x^2}{5}(1-x^2)^{\frac{5}{2}} - \frac{2(1-x^2)^{\frac{7}{2}}}{35} + C.$.

(b) Let $u = e^x$. Then $du = e^x dx$ and $\int \frac{e^x}{(e^x+1)^2(e^{2x}+1)} dx = \int \frac{1}{(u+1)^2(u^2+1)} du.$

The function $\frac{1}{(u+1)^2(u^2+1)}$ has a partial fraction decomposition of the form

$$\frac{1}{(u+1)^2(u^2+1)} = \frac{a}{u+1} + \frac{b}{(u+1)^2} + \frac{cu+d}{u^2+1}.$$

Multiplying by $(u+1)^2(u^2+1)$, we have

$$
\begin{aligned}
1 =& a(u+1)(u^2+1) + b(u^2+1) + (cu+d)(u+1)^2 \\
=& a(u^3+u+u^2+1) + bu^2 + b + (cu+d)(u^2+2u+1) \\
=& au^3 + au^2 + au + a + bu^2 + b + cu^3 + 2cu^2 + cu + du^2 + 2du + d \\
=& (a+c)u^3 + (a+b+2c+d)u^2 + (a+c+d)u + a + b + d
\end{aligned}
$$

Comparing the coefficient, we have $a+c=0, a+b+2c+d=0, a+c+d=0, a+b+d=1$. From $a+c=0$, we have $c=-a$. From $a+c+d=0$ and $a+c=0$, we have $d=0$. From $a+b+d=1$ and $d=0$, we have $b=1-a$. From $a+b+2c+d=0$, we have $a+1-a-2a+0=0$, $2a=1$ and $a=\frac{1}{2}$. Thus $b=1-a=1-\frac{1}{2}=\frac{1}{2}$, $c=-a=-\frac{1}{2}$.

$$\frac{1}{(u+1)^2(u^2+1)} = \frac{1}{2}\frac{1}{u+1} + \frac{1}{2}\frac{1}{(u+1)^2} - \frac{1}{2}\frac{u}{u^2+1}.$$

Remark: one can also plug in $u=-1$ first to get $b=\frac{1}{2}$ first and then determine other coefficients accordingly.

Thus

$$\int \frac{1}{(u+1)^2(u^2+1)}du = \int \frac{1}{2}\frac{1}{u+1} + \frac{1}{2}\frac{1}{(u+1)^2} - \frac{1}{2}\frac{u}{u^2+1}du$$
$$= \frac{1}{2}\ln|u+1| - \frac{1}{2}\frac{1}{u+1} - \frac{1}{4}\ln|u^2+1| + C$$

and

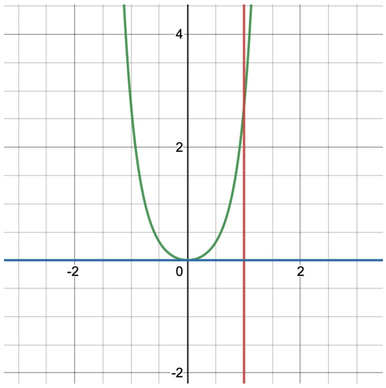
$$\int \frac{e^x}{(e^x+1)^2(e^{2x}+1)}dx = \frac{1}{2}\ln|e^x+1| - \frac{1}{2}\frac{1}{e^x+1} - \frac{1}{4}\ln|e^{2x}+1| + C.$$

1 point for the correct substitution $u = e^x$ and $du = e^x dx$,
1 point for getting $\int \frac{e^x}{(e^x+1)^2(e^{2x}+1)}dx = \int \frac{1}{(u+1)^2(u^2+1)}du$
1 point for setting up $\frac{1}{(u+1)^2(u^2+1)} = \frac{a}{u+1} + \frac{b}{(u+1)^2} + \frac{cu+d}{u^2+1}$
4 points for getting $\frac{1}{(u+1)^2(u^2+1)} = \frac{1}{2}\frac{1}{u+1} + \frac{1}{2}\frac{1}{(u+1)^2} - \frac{1}{2}\frac{u}{u^2+1}$ (each coefficient 1 point)
2 points for getting $\int \frac{1}{(u+1)^2(u^2+1)}du = \frac{1}{2}\ln|u+1| - \frac{1}{2}\frac{1}{u+1} - \frac{1}{4}\ln|u^2+1| + C$
1 point for the final answer $\int \frac{e^x}{(e^x+1)^2(e^{2x}+1)}dx = \frac{1}{2}\ln|e^x+1| - \frac{1}{2}\frac{1}{e^x+1} - \frac{1}{4}\ln|e^{2x}+1| + C$

3. Find the volume of the solid obtained by rotating the following regions about the specific axes.

(a) (8%) Region bounded by $y = \sqrt{\frac{x}{2}}$ and $y = \frac{x^2}{4}$; rotate about the x-axis.

(b) (8%) Region bounded by $y = x^2 e^{x^2}$, $x = 0$, $x = 1$ and $y = 0$ (See Figure below); rotate about the y-axis.



Solution:

(a) The curves $y = f(x) = \sqrt{\frac{x}{2}}$ and $y = g(x) = \frac{x^2}{4}$ intersect when $\sqrt{\frac{x}{2}} = \frac{x^2}{4}$. This implies $\frac{x}{2} = \frac{x^4}{16}$, $x^4 - 8x = 0$ and $x(x^3 - 8) = 0$. So $x = 0$ and $x = 2$. We can plug in $x = \frac{1}{2}$ and get $f(\frac{1}{2}) = \sqrt{\frac{1}{4}} = \frac{1}{2}$ and $g(\frac{1}{2}) = \frac{1}{16}$. So we know that $\sqrt{\frac{x}{2}} \geq \frac{x^2}{4}$ on $[0, 2]$. The cross section has the shape of a washer (an annular ring) with inner radius $\frac{x^2}{4}$ and outer radius $\sqrt{\frac{x}{2}}$. The area of the cross-sectional is $A(x) = \pi(\sqrt{\frac{x}{2}})^2 - \pi(\frac{x^2}{4})^2 = \pi(\frac{x}{2} - \frac{x^4}{16})$.

So the volume is $\int_0^2 A(x)dx = \int_0^2 \pi(\frac{x}{2} - \frac{x^4}{16})dx = \pi(\frac{x^2}{4} - \frac{x^5}{80})\Big|_0^2 = \pi(\frac{4}{4} - \frac{32}{80}) = \frac{3\pi}{5}$.

2 points for finding the intersecting x-coordinate $x = 0$ and $x = 2$,

1 point for showing or explaining $\sqrt{\frac{x}{2}} \geq \frac{x^2}{4}$ on $[0, 2]$

2 points for finding area of cross section $A(x) = \pi(\frac{x}{2} - \frac{x^4}{16})$

1 points for setting up volume $\int_0^2 A(x)dx = \int_0^2 \pi(\frac{x}{2} - \frac{x^4}{16})dx$

2 points for the definite integral final answer $\int_0^2 \pi(\frac{x}{2} - \frac{x^4}{16})dx = \pi(\frac{x^2}{4} - \frac{x^5}{80})\Big|_0^2 = \pi(\frac{4}{4} - \frac{32}{80}) = \frac{3\pi}{5}$

(b) We can use the shell method to find the volume. So the volume is $\int_0^1 2\pi x \cdot x^2 e^{x^2} dx = 2\pi \int_0^1 x^3 e^{x^2} dx$.

We find $\int x^3 e^{x^2} dx$ first. We write $\int x^3 e^{x^2} dx$ as $\int x^2 x e^{x^2} dx$. Let $u = x^2$ and $dv = xe^{x^2} dx$. Then $du = 2x$ and $v = \int xe^{x^2} dx = \frac{e^{x^2}}{2}$. Using integration by parts, we get $\int x^2 x e^{x^2} dx = x^2 \frac{e^{x^2}}{2} - \int \frac{e^{x^2}}{2} \cdot 2x dx = \frac{x^2 e^{x^2}}{2} - \int xe^{x^2} dx = \frac{x^2 e^{x^2}}{2} - \frac{e^{x^2}}{2} + C$.

Thus $2\pi \int_0^1 x^3 e^{x^2} dx = 2\pi(\frac{x^2 e^{x^2}}{2} - \frac{e^{x^2}}{2})|_0^1 = \pi(x^2 e^{x^2} - e^{x^2})|_0^1 = \pi(e - e - (0 - 1)) = \pi \cdot 1 = \pi$.

So the volume is π.

4 points for setting up the volume formula $2\pi \int_0^1 x^3 e^{x^2} dx$,

2 points for indefinite integral $\int x^3 e^{x^2} dx = \frac{x^2 e^{x^2}}{2} - \frac{e^{x^2}}{2} + C$

2 points for finding the definite integral Volume= $2\pi \int_0^1 x^3 e^{x^2} dx = 2\pi(\frac{x^2 e^{x^2}}{2} - \frac{e^{x^2}}{2})|_0^1 = \pi(x^2 e^{x^2} - e^{x^2})|_0^1 = \pi(e - e - (0 - 1)) = \pi \cdot 1 = \pi$

4. The latest model of XPhone has just released. It is very popular that there is a shortage in supply.

(a) The XPhone is priced at $p = D(q) = 64(4 + 2^{-(\frac{q-2}{20})})$ dollars when q-thousand units are demanded. While the supply function is given by $p = S(q) = 256 + 4^{\frac{q}{10}}$, only $q = 10$-thousand units can be supplied.

(i) (6%) Find the total surplus in this case.

(ii) (2%) How many additional units of XPhone need to be supplied to maximize the total surplus ?

Reminder. The total surplus (TS) is defined by $\text{TS}(\bar{q}) = \int_0^{\bar{q}} [D(q) - S(q)]\, dq$.

(b) The waiting time for customers to receive their new XPhone is a continuous random variable whose probability density function equals

$$f(x) = \begin{cases} 0 & \text{if } x < 0 \\ \dfrac{\ln 2}{5} \cdot 2^{-x/5} & \text{if } x \geq 0 \end{cases} \quad (x \text{ in days}).$$

(i) (8%) What is probability for a customer to wait for more than a week ?

(ii) (6%) The newly released earphone 'XPod' is also highly demanded and its waiting time is a random variable given by $Y = 0.5X + 1$. Find the probability density function for Y.

Solution:

(a) (i) The total surplus is

$$\underbrace{\int_0^{10}}_{(1M)} \underbrace{64 \cdot 2^{-(\frac{q-2}{20})} - 4^{\frac{q}{10}}}_{(2M)}\, dq = \left[\underbrace{\frac{-1280}{\ln 2} \cdot 2^{-(\frac{q-2}{20})}}_{(1M)} - \underbrace{\frac{10}{\ln 4} \cdot 4^{\frac{q}{10}}}_{(1M)} \right]_{q=0}^{q=10}$$

$$= \underbrace{-\frac{1280 \cdot 2^{-\frac{2}{5}}}{\ln 2} - \frac{40}{\ln 4} + \frac{1280 \cdot 2^{\frac{1}{10}}}{\ln 2} + \frac{10}{\ln 4}}_{(1M)}$$

$$= \frac{1280(2^{\frac{1}{10}} - 2^{-\frac{2}{5}}) - 15}{\ln 2}$$

Marking scheme :

- 1M for correct integration limits
- 2M for correct integrand
- 1M for correct anti-derivative of $2^{-\frac{q-2)}{20}}$
- 1M for correct anti-derivative of $4^{q/10}$
- 1M for correct answer

(ii) The total surplus is maximized at equilibrium quantity $q^\star$ at which the quantity demanded and supplied agree. Equate

$$\underbrace{64(4 + 2^{-\left(\frac{q^\star - 2}{20}\right)}) = 256 + 4^{\frac{q^\star}{10}}}_{(1M)} \Rightarrow q^\star = \underbrace{24.4}_{(1M)}$$

So $24.4 - 10 = 14.4$-thousand units need to be produced additionally. Marking scheme :

- 1M for setting $D(q) = S(q)$
- 1M for correct answer

(b) (i) The required probability equals to

$$\underbrace{\int_7^\infty}_{(2M)} \underbrace{\frac{\ln 2}{5} \cdot 2^{-x/5}}_{(1M)}\, dx \overset{\text{def}}{=} \underbrace{\lim_{t\to\infty} \int_7^t \frac{\ln 2}{5} \cdot 2^{-x/5}\, dx}_{(1M)} = \lim_{t\to\infty} \left[\underbrace{-2^{-x/5}}_{(2M)} \right]_7^t = \lim_{t\to\infty} (-2^{-t/5} + 2^{-7/5}) = \underbrace{\frac{1}{2^{7/5}}}_{(2M)}$$

Marking scheme :

- 2M for correct integration limits
- 1M for correct integrand

- 1M for definition of improper integral
- 2M for correct antiderivative of $f(x)$
- 2M for correct answer

(ii) The distribution function of Y is (for $y \geq 1$)

$$\underbrace{F_Y(y) = \mathbb{P}(Y \leq y)}_{(1M)} = \underbrace{\mathbb{P}(X \leq 2y-2) = \int_0^{2y-2} \frac{\ln 2}{5} \cdot 2^{-x/5}\, dx}_{(1M)}$$

By differentiating the distribution function, we obtain the density functiin of Y :

$$f_Y(y) = \begin{cases} 0 & \text{if } y < 1 \qquad (1M) \\ \frac{2\ln 2}{5} \cdot 2^{-(2y-2)/5} & \text{if } y \geq 1 \qquad (3M) \end{cases} \quad (y \text{ in days}).$$

Marking scheme :

- 1M for definition of probability distribution function
- 1M for transforming into $\mathbb{P}(X \leq 2y-2)$
- 1M for specifying $f_Y(y) = 0$ for $y < 1$ (or $y \leq 1$)
- 3M for correct density function when $y \geq 1$.

5. Solve, for $y = y(x)$, the following differential equations.

(a) (10%) $\dfrac{dy}{dx} = x^2y^2 - x^2$ with $y(0) = 2$.

(b) (10%) $\dfrac{dy}{dx} = x - 2y + 1$ with $y(0) = \dfrac{1}{8}$.

Solution:

(a) We rewrite the differential equation as $y' = (y^2-1)x^2$. Case 1: $y = \pm 1$. $y' = 0$. Since $y(0) = 2$, it is impossible that the case occurs.
Case 2: $y \neq \pm 1$. It implies that

$$\begin{aligned}
&\int \frac{dy}{y^2-1} = \int x^2 dx (1 \text{ point})\\
\Rightarrow\ & \frac{1}{2}\int \frac{1}{y-1} - \frac{1}{y+1} dy = \frac{1}{3}x^3 (1 \text{ point})\\
\Rightarrow\ & \frac{1}{2}\ln\left|\frac{y-1}{y+1}\right| = \frac{1}{3}x^3 + C(2 \text{ points})\\
\Rightarrow\ & \frac{y-1}{y+1} = Ae^{\frac{2x^3}{3}} (2 \text{ points})\\
\Rightarrow\ & y(x) = \frac{1+Ae^{\frac{2x^3}{3}}}{1-Ae^{\frac{2x^3}{3}}} \text{ where } A \in \mathbb{R}.(1 \text{ point})
\end{aligned}$$

Since $y(0) = 2$, $\dfrac{1+A}{1-A} = 2 \Rightarrow A = \dfrac{1}{3}$.(2 points)
Hence the general solution of the differential equation is

$$y(x) = \frac{1+\frac{1}{3}e^{\frac{2x^3}{3}}}{1-\frac{1}{3}e^{\frac{2x^3}{3}}} = \frac{3+e^{\frac{2x^3}{3}}}{3-e^{\frac{2x^3}{3}}}.(1 \text{ point})$$

(b) The integration factor $I(x)$ is

$$I(x) = e^{\int 2dx} = e^{2x}.(2 \text{ points})$$

Multiply both sides of $y' + 2y = x + 1$ by e^{2x}. We have that

$$\begin{aligned}
&(e^{2x}y)' = e^{2x}(x+1)(1 \text{ point})\\
\Rightarrow\ & e^{2x}y = \int e^{2x}(x+1)dx(1 \text{ point})\\
\Rightarrow\ & e^{2x}y = \int xe^{2x}dx + \int e^{2x}dx\\
\Rightarrow\ & e^{2x}y = \frac{x}{2}e^{2x} - \int \frac{1}{2}e^{2x}dx + \int e^{2x}dx(2 \text{ points})\\
\Rightarrow\ & e^{2x}y = \frac{x}{2}e^{2x} + \frac{1}{4}e^{2x} + C(1 \text{ point})\\
\Rightarrow\ & y = \frac{x}{2} + \frac{1}{4} + Ce^{-2x}.(1 \text{ point})
\end{aligned}$$

Since $y(0) = \dfrac{1}{4} + C = \dfrac{1}{8}$, $C = \dfrac{-1}{8}$(1 point). Hence $y = \dfrac{x}{2} + \dfrac{1}{4} - \dfrac{1}{8}e^{-2x}$ (1 point).

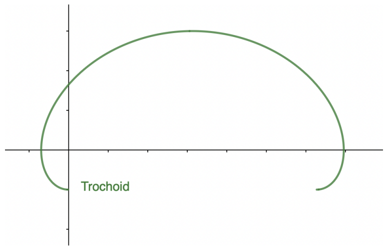
6. The parametric equations of a *trochoid* (see figure) are given by

$$x = \theta - 2\sin(\theta) \text{ and } y = 1 - 2\cos(\theta) \text{ where } 0 \le \theta \le 2\pi.$$

(a) (8%) Find the equation of the tangent to the curve at $\theta = \frac{\pi}{4}$.

(b) (2%) Find the values of θ at which the curve passes the x-axis.

(c) (6%) Find the area of the region under the curve and above the x–axis.



Solution:

(a) We have that $\frac{dx}{d\theta} = 1 - 2\cos(\theta)$ (1 point) and $\frac{dy}{d\theta} = 2\sin(\theta)$ (1 point). We have that

$$\frac{dy}{dx} = \frac{2\sin(\theta)}{1-2\cos(\theta)}.\text{(2 points)}$$

Thus $\frac{dy}{dx}|_{\theta=\pi/4} = \frac{2\sin(\theta)}{1-2\cos(\theta)}|_{\theta=\pi/4} = \frac{\sqrt{2}}{1-\sqrt{2}} = -(\sqrt{2}+2)$(2 points). Hence the tangent line is $y - (1 - 2\cos(\pi/4)) = -(\sqrt{2}+2)(x - \pi/4 + 2\sin(\pi/4) \Rightarrow y - 1 + \sqrt{2} = -(\sqrt{2}+2)(x - \pi/4 + \sqrt{2})$(2 points).

(b) The values of θ at which the curve passes the x-axis are $1 - 2\cos(\theta) = 0$(1 point). It implies that $\theta = \frac{\pi}{3}$(0.5 point) and $\theta = \frac{5\pi}{3}$(0.5 point).

(c) The area is

$$\begin{aligned}
A &= \int y\,dx \\
&= \int_{\frac{\pi}{3}}^{\frac{5\pi}{3}} (1-2\cos\theta)d(\theta - 2\sin(\theta))\text{(1 point)} \\
&= \int_{\frac{\pi}{3}}^{\frac{5\pi}{3}} (1-2\cos\theta)(1-2\cos\theta)d\theta\text{(1 point)} \\
&= \int_{\frac{\pi}{3}}^{\frac{5\pi}{3}} (1-4\cos\theta+4\cos^2\theta)d\theta\text{(1 point)} \\
&= [\theta - 4\sin\theta]_{\frac{\pi}{3}}^{\frac{5\pi}{3}} + \int_{\frac{\pi}{3}}^{\frac{5\pi}{3}} 2\cos(2\theta) + 2d\theta\text{(1 point)} \\
&= \frac{4\pi}{3} + 4\sqrt{3} + [2\theta + \sin(2\theta)]_{\frac{\pi}{3}}^{\frac{5\pi}{3}}\text{ (1 point)} \\
&= \frac{4\pi}{3} + 4\sqrt{3} + \frac{8\pi}{3} - \sqrt{3} = 4\pi + 3\sqrt{3}\text{(1 point)}.
\end{aligned}$$