1. Evaluate

   (a) (8%) $\displaystyle\int_{2\sqrt{2}}^{4} \frac{dx}{x^2(x^2-4)^{3/2}}$

   (b) (8%) $\displaystyle\int_{-1}^{2} x^2 \ln(x+2)\ dx$

Solution:

(a) Let $x = 2\sec\theta$, $0 \le \theta < \pi/2$ or $\pi \le \theta < 3\pi/2$. Then $dx = 2\sec\theta\tan\theta\ d\theta$, $(x^2-4)^{1/2} = 2\tan\theta$. The integration limits for θ can be found by $2\sec\theta = 4$ and $2\sec\theta = 2\sqrt{2}$, hence $\theta = \pi/3$ and $\theta = \pi/4$.

$$\int_{2\sqrt{2}}^{4} \frac{dx}{x^2(x^2-4)^{3/2}} = \frac{1}{2^4}\int_{\pi/4}^{\pi/3} \frac{d\theta}{\sec\theta\tan^2\theta} = \frac{1}{16}\int_{\pi/4}^{\pi/3} \frac{\cos^3\theta}{\sin^2\theta}\ d\theta = \frac{1}{16}\int_{\pi/4}^{\pi/3}\left(\frac{\cos\theta}{\sin^2\theta} - \cos\theta\right)\ d\theta$$

$$= \left[\frac{-\csc\theta - \sin\theta}{16}\right]_{\pi/4}^{\pi/3} = \frac{-\frac{2}{\sqrt{3}} - \frac{\sqrt{3}}{2} + \sqrt{2} + \frac{1}{\sqrt{2}}}{16} = \frac{9\sqrt{2} - 7\sqrt{3}}{96}$$

□

More on (a): Antiderivative

$$\int \frac{dx}{x^2(x^2-4)^{3/2}} = \frac{-\csc\theta - \sin\theta}{16} + C = \frac{-\frac{x}{\sqrt{x^2-4}} - \frac{\sqrt{x^2-4}}{x}}{16} + C = \frac{2-x^2}{8x\sqrt{x^2-4}} + C$$

Substitution $u = x^2 - 4$

$$\int_{2\sqrt{2}}^{4} \frac{dx}{x^2(x^2-4)^{3/2}} = \int_{4}^{12} \frac{du}{2u^{3/2}(u^2+4)^{3/2}}$$

Substitution $u = \sqrt{x^2-4}$

$$\int_{2\sqrt{2}}^{4} \frac{dx}{x^2(x^2-4)^{3/2}} = \int_{2}^{2\sqrt{3}} \frac{du}{u^2(u^2+4)^{3/2}}$$

(b) Integration by parts.

$$\int_{-1}^{2} x^2\ln(x+2)\ dx = \frac{1}{3}\int_{-1}^{2}\ln(x+2)\ d(x^3) = \left[\frac{x^3\ln(x+2)}{3}\right]_{-1}^{2} - \frac{1}{3}\int_{-1}^{2}\frac{x^3}{x+2}\ dx$$

Long division.

$$= \frac{8}{3}\ln 4 - \frac{1}{3}\int_{-1}^{2}\left(x^2 - 2x + 4 - \frac{8}{x+2}\right)\ dx = \frac{8}{3}\ln 4 - \frac{1}{3}\left[\frac{x^3}{3} - x^2 + 4x - 8\ln|x+2|\right]_{-1}^{2}$$

$$= \frac{8}{3}\ln 4 - \frac{1}{3}\left[\left(\frac{8}{3} - 4 + 8 - 8\ln 4\right) - \left(\frac{-1}{3} - 1 - 4\right)\right] = \frac{16\ln 4}{3} - 4$$

□

More on (b): Antiderivative

$$\int x^2\ln(x+2)\ dx = \frac{(x^3+8)\ln(x+2)}{3} - \frac{x^3 - 3x^2 + 12x}{9} + C$$

Substitution $u = x + 2$

$$\int_{-1}^{2} x^2\ln(x+2)\ dx = \int_{1}^{4}(u-2)^2\ln u\ du = \left[\frac{(u-2)^3\ln u}{3}\right]_{1}^{4} - \int_{1}^{4}\frac{u^3 - 6u^2 + 12u - 8}{3u}\ du$$

Substitution $u = \ln(x+2)$

$$\int_{-1}^{2} x^2\ln(x+2)\ dx = \int_{0}^{\ln 4} ue^u(e^u-2)^2\ du = \int_{0}^{\ln 4}(ue^{3u} - 4ue^{2u} + 4ue^u)\ du$$

Grading scheme:

For (a):
(3 pts) for correctly applying a trig-sub. Does not include finding the angles.
(3 pts) for correctly evaluating a trig-integral. They can still get points here even if they mess up the trig-sub.
(2 pts) Notation, convert the bounds, evaluation, and answer. Basically (-1 pt) for each error you find.
For (b):
(3 pts) for correctly applying an integration by parts.
(2 pts) for correctly dealing with the rational function. Mistakes in long division would subtract from here.
(3 pts) Notation, evaluation, and answer. Basically (-1 pt) for each error you find.

2. (12%) Compute the following indefinite integral.

$$\int \frac{1}{\sqrt{x} - \sqrt[3]{x} + \sqrt[6]{x} - 1} dx$$

Solution:

By setting $u = \sqrt[6]{x}$, we have $x = u^6$ $(u \geqslant 0)$ and

$$\int \frac{1}{\sqrt{x} - \sqrt[3]{x} + \sqrt[6]{x} - 1} dx = \int \frac{6u^5}{u^3 - u^2 + u - 1} du = \int \left(6u^2 + 6u + \frac{6u}{u^3 - u^2 + u - 1}\right) du.$$

Note that $u^3 - u^2 + u - 1 = (u-1)(u^2+1)$. Suppose that numbers A, B, and C are such that

$$\frac{6u}{u^3 - u^2 + u - 1} = \frac{A}{u-1} + \frac{Bu + C}{u^2 + 1}, \tag{1}$$

i.e.

$$6u = (A+B)u^2 + (-B+C)u + A - C.$$

Thus, $C = A = -B$ and $6 = -B + C = 2C$. This implies that $C = 3$, $A = 3$, and $B = -3$, and hence

$$\begin{aligned}
\int \frac{1}{\sqrt{x} - \sqrt[3]{x} + \sqrt[6]{x} - 1} dx &= \int (6u^2 + 6u) du + \int \frac{6u\, du}{u^3 - u^2 + u - 1} \\
&= \int (6u^2 + 6u) du + 3\int \frac{du}{u-1} - 3\int \frac{u\, du}{u^2+1} + 3\int \frac{du}{u^2+1} \\
&= 2u^3 + 3u^2 + 3\ln|u-1| - \frac{3}{2}\ln(u^2+1) + 3\arctan u + C \\
&= 2\sqrt{x} + 3\sqrt[3]{x} + 3\ln|\sqrt[6]{x} - 1| - \frac{3}{2}\ln(\sqrt[3]{x} + 1) + 3\arctan(\sqrt[6]{x}) + C.
\end{aligned}$$

給分原則：本題大致分為三個部分：

(I) (4分) 利用變數代換將問題轉化為有理函數的積分：

(1) 選擇適當的變數變換（例如令$x = u^6$）佔2分；

(2) 將dx正確轉換（若取$x = u^6$則$dx = 6u^5 du$）佔2分。

(II) (4分) 做有理函數的部分分式展開：

(1) 先將分子除以分母求商（上述解答過程中的$6u^2 + 6u$），並將剩餘部分設定為正確的形式 $\frac{A}{u-1} + \frac{Bu+c}{u^2+1}$ 佔2分。

(2) 正確計算A、B與C佔2分。此部分無部分分數。

(III) (4分) 求出 (II) 所得之部分分式各項的不定積分，分為以下四部分，皆不給部分分數，遺漏積分常數視為算錯：

(1) 多項式部分（$6u^2 + 6u$）的不定積分佔1分；

(2) 正確求出$\int \frac{u}{1+u^2} du$佔1分；

(3) 正確求出$\int \frac{1}{1+u^2} du$佔1分；

(4) 正確代換回以x為變數的函數佔1分。

部分分數：

(II-2) 如果商算錯但是部分分式的形式設定正確，可得部分分數1分。

- 如果變數代換後的有理函數（例如上述的$\frac{6u}{u^3 - u^2 + u - 1}$）寫錯，但是分母為次數至少為3的多項式，那麼(I)得0分，但是(II)與(III)的部分依所得（錯誤的）有理函數按照前述標準給分。
-) 如果(II)的部分分式寫錯，(III)的部分依所得（錯誤的）部分分式按照前述標準給分。

3. Determine whether the improper integral is convergent or divergent. Provide your reasoning **in detail**. Evaluate the integral if it is convergent.

   (a) (6%) $\int_0^1 \frac{2+\sin x}{x\ln x}\,dx$

   (b) (6%) $\int_0^\infty e^{-2x}\sin(3x)\,dx$

Solution:

(a) The final answer is divergent. The integral is improper at both $x = 0$ and $x = 1$.

$$-1 \le \sin x \le 1 \;\Rightarrow\; 1 \le 2+\sin x \le 3 \;\Rightarrow\; \frac{1}{x\ln x} \ge \frac{2+\sin x}{x\ln x} \ge \frac{3}{x\ln x}$$

since $\ln x < 0$.

To apply comparison theorem we would need to consider $\int_0^1 \frac{2+\sin x}{x(-\ln x)}\,dx$, along with

$$0 \le \frac{1}{x(-\ln x)} \le \frac{2+\sin x}{x(-\ln x)}$$

$$\lim_{a\to 0^+}\int_a^{1/2} \frac{1}{x(-\ln x)}\,dx = \lim_{a\to 0^+}\left(-\ln|\ln(1/2)| + \ln|\ln a|\right) = \infty$$

The limit does not exist. The improper integral $\int_0^{1/2}\frac{1}{x(-\ln x)}\,dx$ is divergent. Therefore the improper integral $\int_0^1 \frac{1}{x(-\ln x)}\,dx$ and hence $\int_0^1 \frac{2+\sin x}{x\ln x}\,dx$ is divergent.

(b) The final answer is convergent. That means we need to evaluate it.

$$\int e^{-2x}\sin(3x)\,dx = \frac{-1}{2}e^{-2x}\sin(3x) + \frac{3}{2}\int e^{-2x}\cos(3x)\,dx$$
$$= \frac{-1}{2}e^{-2x}\sin(3x) - \frac{3}{4}e^{-2x}\cos(3x) - \frac{9}{4}\int e^{-2x}\sin(3x)\,dx$$

Hence

$$\int e^{-2x}\sin(3x)\,dx = \frac{-1}{13}\left(2e^{-2x}\sin(3x) + 3e^{-2x}\cos(3x)\right)$$

Since $\sin(3x)$ and $\cos(3x)$ are bounded, we can use squeeze theorem to get

$$\lim_{t\to\infty}\frac{-1}{13}\left(2e^{-2t}\sin(3t) + 3e^{-2t}\cos(3t)\right) = 0$$

Therefore

$$\int_0^\infty e^{-2x}\sin(3x)\,dx = \frac{3}{13}$$

(Method 2)

Since $|\sin 3x| \le 1\ \forall x\in\mathbb{R} \Rightarrow |e^{-2x}\sin 3x| \le |e^{-2x}|\ \forall x \in \mathbb{R}$

$\Rightarrow \int_0^\infty e^{-2x}dx = \lim_{a\to\infty} -\frac{1}{2}e^{-2x}\Big|_0^a = \lim_{a\to\infty} -\frac{1}{2}e^{-2a} + \frac{1}{2} = \frac{1}{2} \Rightarrow$ conv

By Ci, $\int_0^\infty e^{-2x}\sin 3x dx$ conv

consider $\mathcal{L}\{\sin 3t\} = \frac{3}{s^2+9} = \int_0^\infty e^{-st}\sin 3t dt$

By assigning $s = 2$, we can have $\int_0^\infty e^{-2t}\sin 3t dt = \frac{3}{s^2+9}\Big|_{s=2} = \frac{3}{13}$

Grading scheme:

For (a):

(3 pts) for showing an improper integral diverges via the limit notation.

(3 pts) for correctly using the comparison theorem. No need to convert to positive but they need to know the inequality is the other way around for negative functions, (-1 pt) if you can tell they didn't see $\ln x$ is negative.

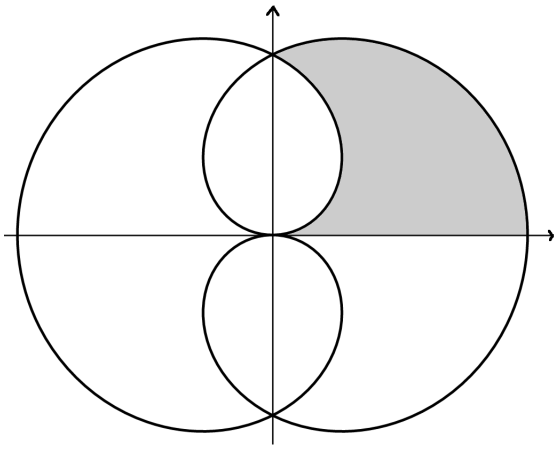
For (b):

(4 pts) for obtaining the antiderivative.

(2 pts) for the limit notation, the conclusion, and the answer.

They can get (4 pts) if they could not evaluate the integral but used other methods to prove convergence.

4. The curve in the xy-plane described by the equation $r = \cos(\theta/2)$ in polar coordinates is shown in the following picture:



(a) (8%) Find all points (in terms of their xy-coordinates) where the above curve has a vertical tangent line.

(b) (8%) Compute the area of the shaded region.

Solution:

(a) Let $P(\theta) = (x(\theta), y(\theta))$ be the point on the curve corresponding to θ. The condition for vertical tangent line is essentially $x'(\theta) = 0$ (provided $y'(\theta) \neq 0$). Note that

$$x(\theta) = \cos(\theta/2)\cos\theta,$$

and hence

$$\begin{aligned} x'(\theta) &= -\frac{1}{2}\sin(\theta/2)\cos\theta - \cos(\theta/2)\sin\theta = -\frac{1}{2}\sin(\theta/2)\big(\cos\theta + 4\cos^2(\theta/2)\big) \\ &= -\frac{1}{2}\sin(\theta/2)(3\cos\theta + 2). \end{aligned}$$

Similarly,

$$\begin{aligned} y'(\theta) &= -\frac{1}{2}\sin(\theta/2)\sin\theta - \cos(\theta/2)\cos\theta = -\cos(\theta/2)\big(\sin^2(\theta/2) - \cos\theta\big) \\ &= -\cos(\theta/2)(1 - \frac{3}{2}\cos\theta). \end{aligned}$$

It is not hard to see that $x'(\theta)$ and $y'(\theta)$ can not be zero simultaneously. The condition $x'(\theta) = 0$ is equivalent to that either $\sin(\theta/2) = 0$ or $\cos\theta = -2/3$. The first situation corresponds to the points $(1, 0)$ and $(-1, 0)$. In the second situation,

$$\cos^2(\theta/2) = \frac{1 - \frac{2}{3}}{2} = \frac{1}{6} \quad \text{and} \quad \sin^2(\theta/2) = \frac{5}{6},$$

and hence $\sin\theta = \pm\dfrac{\sqrt{5}}{3}$. In summary, the corresponding points are

$$\pm\sqrt{\frac{1}{6}}\left(\frac{-2}{3}, \pm\frac{\sqrt{5}}{3}\right) \quad \text{(4 combinations in total).}$$

<u>給分原則</u>：

(1) 顯示出考慮$x'(\theta) = 0$這個方程估2分。

(2) 正確寫出$x'(\theta)$或與其等價物件估2分。

(3) 正確求出$x'(\theta) = 0$的所有解估2分。

(4) 找出對應的xy-坐標點估2分。

部分分數： (4) 的答案含有6個點，寫對的點數若達到4個但不到6個，可得1分，少於4個則得0分。

(b) We denote D_1 (resp. D_2) be the region swiped by the segment $OP(\theta)$ with $\theta \in [0, \frac{\pi}{2}]$ (resp. $\theta \in [\frac{\pi}{2}, \frac{3\pi}{2}]$). Then

$$
\begin{aligned}
\text{the area of the shaded region } &= \text{ the area of } D_1 - \text{ the area of } D_2 \\
&= \frac{1}{2}\int_0^{\frac{\pi}{2}} \cos^2(\theta/2)d\theta - \frac{1}{2}\int_{\pi}^{\frac{3\pi}{2}} \cos^2(\theta/2)d\theta \\
&= \frac{1}{2}\left(\int_0^{\frac{\pi}{2}} \frac{1+\cos\theta}{2}d\theta - \int_{\pi}^{\frac{3\pi}{2}} \frac{1+\cos\theta}{2}d\theta\right) \\
&= \frac{1}{2}\left(\frac{\theta+\sin\theta}{2}\Big|_0^{\frac{\pi}{2}} - \frac{\theta+\sin\theta}{2}\Big|_{\pi}^{\frac{3\pi}{2}}\right) = \frac{1}{2}.
\end{aligned}
$$

給分原則：

(1) 列出代表陰影部分面積的積分差佔4分。如果是採極座標，計算過程中必須能清楚看出被積函數是 $(\frac{r^2}{2} =)\frac{\cos^2\frac{\theta}{2}}{2}$ ，各積分的積分範圍也要正確。如果是採參數化曲線，各積分必須正確表示成yx'或xy'在適當範圍的積分。

(2) 正確算出積分差。兩個積分，各佔2分。

部分分數：(1)的答案若不正確，下列三種情況將給予部分分數：

(i) 列式錯誤（各積分的上下限或被積函數有誤），但是有表明考慮的是正確的兩塊區域的面積差（例如有畫圖或文字說明），可得1分。

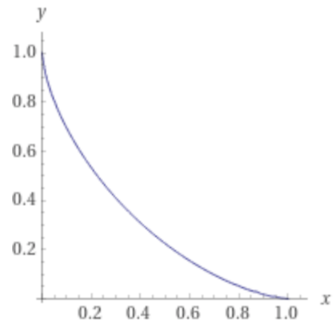
(ii) 各積分的上下限都寫對，但被積函數出錯，可得1分。

(iii) 各積分被積函數都寫對，但上下限出錯，可得1分。

若是(1)所列積分的上下限錯誤，(2)的部分依錯誤的範圍求積分時，不重複扣分。

5. Consider the solid described by $x^{2/3} + y^{2/3} + z \le 2$, $x, y, z \ge 0$.

(a) (8%) Let $x(t) = K\cos^3 t$, $y(t) = K\sin^3(t)$, with $0 \le t \le \pi/2$ and K a positive constant. Find the area of the region in the first quadrant enclosed by the curve, the x-axis, and the y-axis.



(b) (4%) Find the volume of the solid. (Consider cross-sectional area for fixed z-value)

Solution:

(a) The area of the region is

$$\int_0^K y\,dx = -\int_0^{\frac{\pi}{2}} K\sin^3 t\; d\left(K\cos^3 t\right) \qquad \textbf{(2 points)}$$

$$= 3K^2 \int_0^{\frac{\pi}{2}} \sin^4 t\cos^2 t\,dt \qquad \textbf{(1 point)}$$

$$= \frac{3}{8}K^2 \int_0^{\frac{\pi}{2}} (1-\cos(2t))\sin^2(2t)\;dt = \frac{3}{8}K^2\left\{\int_0^{\frac{\pi}{2}} \sin^2(2t)\;dt - \int_0^{\frac{\pi}{2}} \sin^2(2t)\cos(2t)\;dt\right\}$$

$$= \frac{3}{8}K^2\left\{\frac{1}{2}\int_0^{\frac{\pi}{2}} (1-\cos(4t))\;dt - \frac{1}{2}\int_0^{\frac{\pi}{2}} \sin^2(2t)\;d\sin(2t)\right\} \qquad \textbf{(3 points)}$$

$$= \frac{3}{8}K^2\left\{\frac{1}{2}\left[t - \frac{1}{4}\sin(4t)\right]_{t=0}^{\frac{\pi}{2}} - \frac{1}{2}\left[\frac{1}{3}\sin^3(2t)\right]_{t=0}^{\frac{\pi}{2}}\right\} = \frac{3}{32}\pi K^2 \qquad \textbf{(2 points)}.$$

(b) The cross-sectional area of the solid in the plane through z and orthogonal to the $z-$axis is bounded by the curve

$$x(t) = (2-z)^{\frac{3}{2}}\cos^3 t,\; y(t) = (2-z)^{\frac{3}{2}}\sin^3 t,\;\; t \in \left[0, \frac{\pi}{2}\right],$$

the $x-$axis, and the $y-$axis, where $z \in [0,2]$. Using (a) with $K = (2-z)^{\frac{3}{2}}$, the cross-sectional area is

$$A(z) = \frac{3}{32}\pi(2-z)^3 \qquad \textbf{(1 point)}.$$

Thus the volume of the solid is

$$V = \int_0^2 A(z)\;dz \qquad \textbf{(1 point)}$$

$$= \frac{3}{32}\pi\int_0^2 (2-z)^3 dz = \frac{3}{32}\pi\left[-\frac{1}{4}(2-z)^4\right]_{z=0}^2 = \frac{3}{8}\pi. \qquad \textbf{(2 points)}$$

6. (10%) Solve the initial value problem

$$(e^x+1)\,\frac{dy}{dx}+y=1,\quad y(0)=0.$$

Solution:

Method 1: The standard form of the linear equation is

$$\frac{dy}{dx}+\frac{1}{e^x+1}\,y=\frac{1}{e^x+1}.\qquad \textbf{(1 point)}$$

Evaluating the indefinite integral

$$\int\frac{1}{e^x+1}\,dx \overset{u=e^x+1}{=} \int\frac{1}{u(u-1)}\,du=\int\left(\frac{1}{u-1}-\frac{1}{u}\right)du$$

$$=\ln\left|\frac{u-1}{u}\right|+c=\ln\left(\frac{e^x}{e^x+1}\right)+c,\qquad \textbf{(3 points)}$$

gives an integrating factor $\frac{e^x}{e^x+1}$. After multiplying both sides by the integrating factor, the equation becomes

$$\frac{d}{dx}\left(\frac{e^x}{e^x+1}\,y\right)=\frac{e^x}{(e^x+1)^2},\qquad \textbf{(3 points)}$$

from which we get

$$\frac{e^x}{e^x+1}\,y=\frac{-1}{e^x+1}+c.\qquad \textbf{(2 points)}$$

Plugging the initial condition $y(0)=0$ yields

$$y=\frac{1}{2}\left(1-e^{-x}\right).\qquad \textbf{(1 point)}$$

Method 2: The equation is separable, i.e.

$$\frac{1}{y-1}\,dy=\frac{-1}{e^x+1}\,dx.\qquad \textbf{(3 points)}$$

Since

$$\int\frac{-1}{e^x+1}\,dx \overset{u=e^x+1}{=} \int\frac{-1}{u(u-1)}\,du=\int\left(\frac{1}{u}-\frac{1}{u-1}\right)du$$

$$=\ln\left|\frac{u}{u-1}\right|+c=\ln\left(1+e^{-x}\right)+c,\qquad \textbf{(3 points)}$$

we get

$$\ln|y-1|=\ln\left(1+e^{-x}\right)+c \;\Leftrightarrow\; |y-1|=e^c\left(1+e^{-x}\right).\qquad \textbf{(2 points)}$$

Plugging the initial condition $y(0)=0$ gives $e^c=\frac{1}{2}$ and hence

$$y-1=\pm\frac{1}{2}\left(1+e^{-x}\right).\qquad \textbf{(1 point)}$$

The condition $y(0)=0$ implies that $y-1\neq\frac{1}{2}\left(1+e^{-x}\right)$. Therefore,

$$y=1-\frac{1}{2}\left(1+e^{-x}\right)=\frac{1}{2}\left(1-e^{-x}\right).\qquad \textbf{(1 point)}$$

7. (01-02班) Solve the differential equations.

(a) (11%) Use the method of variation of parameters: $y'' - 6y' + 9y = e^{3t}$.

(b) (11%) Computing the Laplace transform of $f(t)$ and solve the boundary value problem

$$\frac{d^2y}{dt^2} + y = f(t), \quad 0 \le t \le 2,$$
$$y(0) = y(2) = 0,$$

where $f(t) = 1 - |t-1|$.

Solution:

(a) The auxiliary equation $X^2 - 6X + 9 = 0$ has repeated roots $r = 3$. Let $y_1(t) = te^{3t}$ and $y_2(t) = e^{3t}$ and set

$$y(t) = A(t)y_1(t) + B(t)y_2(t).$$

We need to solve (A, B) for the system of equations:

$$\begin{cases} te^{3t}A' + e^{3t}B' & = & 0 \\ (e^{3t} + 3te^{3t})A' + 3e^{3t}B' & = & e^{3t}. \end{cases} \tag{2}$$

We then have

$$A'(t) = 1 \text{ and } B'(t) = -t,$$

that is,

$$A(t) = t + C_1 \text{ and } B(t) = -\frac{t^2}{2} + C_2,$$

and hence

$$y(t) = \frac{t^2}{2}e^{3t} + C_1te^{3t} + C_2e^{3t}.$$

<u>給分原則</u>：分成兩部分：

(I) 寫出$y_1 = te^{3t}$與$y_2(t) = e^{3t}$佔3分。

(II) 正確求出$A(t)$與$B(t)$佔8分。

部分分數： 關於(II)的部分，下列三種情況將給予部分分數：

(i) 僅列出方程組(2)但沒有正確寫出A'、B'（或表達A、B的積分），可得2分。

(ii) 正確寫出A'、B'（或表達A、B的積分），但是最後沒有將A、B完全正確算出，可得4分。

(iii) 正確寫出A'、B'（或表達A、B的積分），但是兩個積分只算對一個，得6分。

(b) Although the original question only asks us to solve the ODE on $[0, 2]$, we may simply solve it on $[0, \infty)$. First we compute $\mathcal{L}\{f\}$. Note that

$$f(t) = t + \begin{cases} 0 & \text{if } 0 \leqslant t < 1; \\ 2 - 2t & \text{if } t \geqslant 1 \end{cases} = t - 2(t-1)\mathcal{U}(t-1),$$

and hence

$$\mathcal{L}\{f\} = \mathcal{L}\{t\} - 2\mathcal{L}(t-1)\mathcal{U}(t-1) = \frac{1}{s^2} - 2e^{-s} \cdot \frac{1}{s^2}.$$

Applying the Laplace transform to the original ODE, we obtain that

$$(s^2 + 1)\mathcal{L}\{y\} - y(0)s - y'(0) = \mathcal{L}\{f\}, \tag{3}$$

that is,

$$\begin{aligned} \mathcal{L}\{y\} &= y(0)\frac{s}{s^2+1} + y'(0)\frac{1}{s^2+1} + \frac{\mathcal{L}\{f\}}{s^2+1} \\ &= y'(0)\frac{1}{s^2+1} + \frac{1}{s^2(s^2+1)} - 2e^{-s} \cdot \frac{1}{s^2(s^2+1)} \quad \text{since } y(0) = 0, \\ &= y'(0)\frac{1}{s^2+1} + \frac{1}{s^2} - \frac{1}{s^2+1} - 2e^{-s}\left(\frac{1}{s^2} - \frac{1}{s^2+1}\right) \\ &= \mathcal{L}\{y'(0)\sin t + t - \sin t - 2(t - 1 - \sin(t-1))\mathcal{U}(t-1)\}, \end{aligned}$$

and hence

$$y(t) = y'(0)\sin t + t - \sin t - 2(t-1-\sin(t-1))\mathcal{U}(t-1).$$

To determine $y'(0)$, we use the condition that $y(2)=0$:

$$\begin{aligned} 0 = y(2) &= y'(0)\sin 2 + 2 - \sin 2 - 2(2-1-\sin(2-1))\mathcal{U}(2-1) \\ &= y'(0)\sin 2 + 2 - \sin 2 - 2 + 2\sin 1 \\ &= y'(0)\sin 2 - \sin 2 + 2\sin 1, \end{aligned}$$

and hence

$$y'(0) = \frac{\sin 2 - 2\sin 1}{\sin 2} = 1 - \frac{2\sin 1}{2(\sin 1)\cos 1} = 1 - \sec 1.$$

Therefore,

$$\begin{aligned} y(t) &= (1-\sec 1)\sin t + t - \sin t - 2(t-1-\sin(t-1))\mathcal{U}(t-1) \\ &= t - (\sec 1)\sin t - 2(t-1-\sin(t-1))\mathcal{U}(t-1) \\ &= \begin{cases} t - \sec 1 \sin t & \text{if} \quad 0 \leqslant t < 1; \\ 2 - t - (\sec 1)\sin t + 2\sin(t-1) & \text{if} \quad t \geqslant 1. \end{cases} \end{aligned}$$

Some of you might have regarded $f(t)$ as being 0 for $t > 2$:

$$\begin{aligned} f(t) &= \begin{cases} t & \text{if} \quad 0 \leqslant t < 1; \\ 2-t & \text{if} \quad 1 \leqslant t < 2; \\ 0 & \text{if} \quad t \leqslant 2. \end{cases} = t + \begin{cases} 0 & \text{if} \quad 0 \leqslant t < 1; \\ 2-t-t & \text{if} \quad 1 \leqslant t < 2; \\ -t & \text{if} \quad t \leqslant 2. \end{cases} \\ &= t + \begin{cases} 0 & \text{if} \quad 0 \leqslant t < 1; \\ 2-2t & \text{if} \quad t \geqslant 1 \end{cases} + \begin{cases} 0 & \text{if} \quad 0 \leqslant t < 2; \\ -t-(2-2t) & \text{if} \quad t \geqslant 2 \end{cases} \\ &= t - 2(t-1)\mathcal{U}(t-1) + (t-2)\mathcal{U}(t-2). \end{aligned}$$

Then

$$\mathcal{L}\{f\} = \mathcal{L}\{t\} - 2\mathcal{L}(t-1)\mathcal{U}(t-1) = \frac{1}{s^2} - 2e^{-s}\cdot\frac{1}{s^2} + e^{-2s}\cdot\frac{1}{s^2}.$$

Applying the Laplace transform to the original ODE, we obtain that

$$(s^2+1)\mathcal{L}\{y\} - y(0)s - y'(0) = \mathcal{L}\{f\},$$

that is,

$$\begin{aligned} \mathcal{L}\{y\} &= y(0)\frac{s}{s^2+1} + y'(0)\frac{1}{s^2+1} + \frac{\mathcal{L}\{f\}}{s^2+1} \\ &= y'(0)\frac{1}{s^2+1} + \frac{1}{s^2(s^2+1)} - 2e^{-s}\cdot\frac{1}{s^2(s^2+1)} + e^{-2s}\cdot\frac{1}{s^2(s^2+1)} \qquad \text{since} \quad y(0)=0, \\ &= y'(0)\frac{1}{s^2+1} + \frac{1}{s^2} - \frac{1}{s^2+1} - 2e^{-s}\left(\frac{1}{s^2} - \frac{1}{s^2+1}\right) + e^{-2s}\left(\frac{1}{s^2} - \frac{1}{s^2+1}\right) \\ &= \mathcal{L}\left\{ \begin{array}{c} y'(0)\sin t + t - \sin t \\ -2(t-1-\sin(t-1))\mathcal{U}(t-1) \\ +(t-2-\sin(t-2))\mathcal{U}(t-2) \end{array} \right\}, \end{aligned}$$

and hence

$$y(t) = y'(0)\sin t + t - \sin t - 2(t-1-\sin(t-1))\mathcal{U}(t-1) + (t-2-\sin(t-2))\mathcal{U}(t-2).$$

The rest of the computation is essentially the same as the original one above.

<u>給分原則</u>：
(I) （3分）正確求出$\mathcal{L}\{f\}$。
(II) （8分）正確算出$y(t)$佔8分：

(1) 寫出(3)或本質上等價的結論（比方說已將y(0)=0代入或是將方程兩側除上s^2+1）佔2分；

(2) 正確處理了$y(0)\dfrac{s}{s^2+1}$這個項（說它是0或是注意到它是$\mathcal{L}\{y(0)\cos t\}$）佔1分；

(3) 注意到$y'(0)\dfrac{s}{s^2+1} = \mathcal{L}\{y'(0)\sin t\}$佔1分；

(4) 正確獲得下列計算結果佔3分：

$$\begin{aligned}\frac{\mathcal{L}\{f\}}{s^2+1} &= \frac{1}{s^2(s^2+1)} - 2e^{-s}\cdot\frac{1}{s^2(s^2+1)}\\ &= \frac{1}{s^2} - \frac{1}{s^2+1} - 2e^{-s}\left(\frac{1}{s^2} - \frac{1}{s^2+1}\right)\\ &= \mathcal{L}\{t - \sin t - 2(t-1-\sin(t-1))\mathcal{U}(t-1)\}\end{aligned}$$

更精細地說，將$\frac{1}{s^2(s^2+1)}$正確分解為$\frac{1}{s^2} - \frac{1}{s^2+1}$佔1分，而將$e^{-s}\left(\frac{1}{s^2} - \frac{1}{s^2+1}\right)$正確識別為$\mathcal{L}\{(t-1+\sin(t-1))\mathcal{U}(t-1)\}$佔2分。(另一種情況類似。)

(5) 利用邊界值條件正確決定所有常數佔1分。

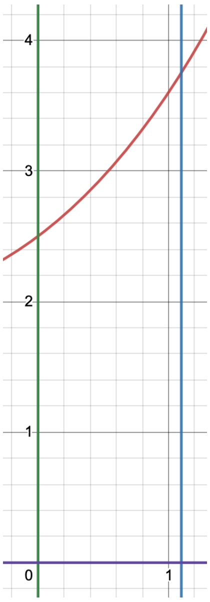
部分分數：(I)的答案若不正確，下列兩種情況將給予部分分數：

(i) 代數錯誤（抄錯符號或係數）以外無誤，可得2分。

(ii) 使用$e^{-as}\mathcal{L}\{f\} = \mathcal{L}\{f(t-a)\mathcal{U}(t-a)\}$ $(a>0)$的方式錯誤，可得1分。

如果在(I)中所得的$\mathcal{L}\{f\}$錯誤，然後(II)中使用這個錯誤的$\mathcal{L}\{f\}$計算時，不重複扣分。若在(II)中決定常數之前$y(t)$的形式錯誤，利用邊界條件決定常數時不重複扣分。

(03-05班) Consider the region R bounded by $y = 2e^{x/2} + \frac{1}{2}e^{-x/2}$, $y = 0$, $x = 0$, and $x = \ln 3$.



(a) (4%) Find the area of R.

(b) (6%) Find the length of the perimeter（周長） of R.

(c) (6%) Find the volume of the solid of revolution obtained by rotating R about the x-axis.

(d) (6%) Find the volume of the solid of revolution obtained by rotating R about the y-axis.

Solution:

(a)

$$\int_0^{\ln 3} \left(2e^{x/2} + \frac{1}{2}e^{-x/2}\right)\, dx = \left[4e^{x/2} - e^{-x/2}\right]_0^{\ln 3} = 4\sqrt{3} - \frac{1}{\sqrt{3}} - 4 + 1 = \frac{11\sqrt{3} - 9}{3}$$

(b) At $x = 0$, $y = \frac{5}{2}$. At $x = \ln 3$, $y = 2\sqrt{3} + \frac{1}{2\sqrt{3}}$.

$$\int_0^{\ln 3} \sqrt{1 + \left(e^{x/2} - \frac{1}{4}e^{-x/2}\right)^2}\, dx = \int_0^{\ln 3} \left(e^{x/2} + \frac{1}{4}e^{-x/2}\right)\, dx = 2\sqrt{3} - \frac{1}{2\sqrt{3}} - 2 + \frac{1}{2} = \frac{11\sqrt{3} - 9}{6}$$

$$\text{Total length} = \left[\frac{5}{2} + \ln 3 + \left(2\sqrt{3} + \frac{1}{2\sqrt{3}}\right) + \frac{11\sqrt{3} - 9}{6}\right]$$

(c)

$$\int_0^{\ln 3} \pi\left(2e^{x/2} + \frac{1}{2}e^{-x/2}\right)^2\, dx = \pi\int_0^{\ln 3}\left(2 + 4e^{x} + \frac{1}{4}e^{-x}\right)\, dx = \pi\left[2x + 4e^{x} - \frac{1}{4}e^{-x}\right]_0^{\ln 3}$$
$$= \pi\left(2\ln 3 + 12 - \frac{1}{12} - 0 - 4 + \frac{1}{4}\right) = \pi\left(2\ln 3 + 8 + \frac{1}{6}\right)$$

(d)

$$\int_0^{\ln 3} 2\pi x\left(2e^{x/2} + \frac{1}{2}e^{-x/2}\right)\, dx = 2\pi\left[x(4e^{x/2} - e^{-x/2})\right]_0^{\ln 3} - 2\pi\int_0^{\ln 3}\left(4e^{x/2} - e^{-x/2}\right)\, dx$$
$$= 2\pi\left[x(4e^{x/2} - e^{-x/2}) - 8e^{x/2} - 2e^{-x/2}\right]_0^{\ln 3} = 2\pi\left[4\sqrt{3}\ln 3 - \frac{\ln 3}{\sqrt{3}} - 8\sqrt{3} - \frac{2}{\sqrt{3}} + 8 + 2\right]$$
$$= \frac{2\pi}{3}\left(30 - 26\sqrt{3} + 11\sqrt{3}\ln 3\right)$$

Grading scheme:

No points if the integral is set up incorrectly.
(-4 pts) for mixing up (c) and (d).
(-2 pts) for each error (notation, computation, copying, integration of each term).
(-2 pts) only once for not simplifying $e^{\ln 3/2} = \sqrt{3}$.
(-2 pts) only once for over-simplifying.