1092 Calculus4 01-05 Final Exam Solution

June 19, 2021

1. (10 pts) Let D be any square in $\mathbb{R}^2$ not containing the origin with side length $a > 0$. Let C be the positively oriented boundary of D. Evaluate

$$\int_C \left(\frac{2x^2+y^2}{\sqrt{x^2+y^2}} + y\right) dx + \left(\frac{xy}{\sqrt{x^2+y^2}} - x\right) dy$$

Solution:

By Green's Theorem

$$\int_C \left(\frac{2x^2+y^2}{\sqrt{x^2+y^2}} + y\right) dx + \left(\frac{xy}{\sqrt{x^2+y^2}} - x\right) dy = \iint_D +\frac{\partial}{\partial x}\left(\frac{xy}{\sqrt{x^2+y^2}} - x\right) - \frac{\partial}{\partial y}\left(\frac{2x^2+y^2}{\sqrt{x^2+y^2}} + y\right) dA$$

至此正確 6分。沒有寫 "Green's Theorem"，扣 1分。
公式寫成 $-\frac{\partial}{\partial x}(\ ) + \frac{\partial}{\partial y}(\ )$，扣 2分。寫成 $+\frac{\partial}{\partial x}(\ ) + \frac{\partial}{\partial y}(\ )$ 或 $-\frac{\partial}{\partial x}(\ ) - \frac{\partial}{\partial y}(\ )$，整題只給 1分，下面不必看。

$$= \iint_D \underbrace{\frac{\partial}{\partial x}\left(\frac{xy}{\sqrt{x^2+y^2}}\right) - \frac{\partial}{\partial y}\left(\frac{2x^2+y^2}{\sqrt{x^2+y^2}}\right)}_{=0} + \frac{\partial}{\partial x}(-x) - \frac{\partial}{\partial y} y \, dA$$

計算過程不必看，得到 0，得 2分

$$= \iint_D -2 \, dA$$ 正負號上方已考慮了。算出 2，得 1分

$= -2\text{Area}(D) = -2a^2$ 此處的 2, a^2 為獨立給分，不管另一塊偏導數計算是否算出 0。

寫出 a^2，得 1分

使用 1個特定的正方形，用 Green's Theorem 或 line integral 算, 得到正確答案, 6分. 算錯, 0分.

2. (14 pts) Evaluate $\iint_S z\,dS$, where S is the part of the paraboloid $z = x^2 + y^2$ that lies under the plane $z = 4$.

Solution:

Method 1. Notice that S is the graph of the function $f(x,y) = x^2 + y^2$ above the disc D of radius 2 centered at origin. (2 points)

In this case, we have

$$dS = \sqrt{1 + f_x^2 + f_y^2}\,dA = \sqrt{1 + (2x)^2 + (2y)^2}\,dA. \quad \text{(4 points)}$$

Therefore,

$$\iint_S z\,dS = \iint_D (x^2 + y^2)\sqrt{1 + (2x)^2 + (2y)^2}\,dA. \qquad (*)$$

Using polar coordinates $(x,y) = r(\cos\theta, \sin\theta)$, we have

$$dA = r dr d\theta, \quad \text{(3 points)}$$
$$(*) = 2\pi \int_0^2 r^2\sqrt{1 + 4r^2} \cdot r\,dr. \quad \text{(1 points)}$$

Here,

$$\begin{aligned}\int_0^2 r^3\sqrt{1+4r^2}\,dr &= \frac{r^2}{12}(1+4r^2)^{3/2}\Big|_0^2 - \frac{1}{6}\int_0^2 r(1+4r^2)^{3/2}\,dr \\ &= \frac{17^{3/2}}{3} - \frac{1}{120}(1+4r^2)^{5/2}\Big|_0^2 \\ &= \frac{17^{3/2}}{3} - \frac{17^{5/2}}{120} + \frac{1}{120}. \quad \text{(4 points; allowing small mistake)}\end{aligned}$$

All together, one obtains $\iint_S z\,dS = 2\pi\Big[\frac{17^{3/2}}{3} - \frac{17^{5/2}}{120} + \frac{1}{120}\Big]$.

Method 2. One uses the parametrization of S:

$$r(\rho,\theta) = \langle x = \rho\cos\theta, y = \rho\sin\theta, z = x^2 + y^2 = \rho^2\rangle,$$
$$0 \le \rho \le 2, 0 \le \theta \le 2\pi. \quad \text{(2 points)}$$

In this case,

$$r_\rho \times r_\theta = \langle -2\rho^2\cos\theta, -2\rho^2\sin\theta, \rho\rangle,$$
(3 points; it is OK if there is sign error)
$$dA = \|r_\rho \times r_\theta\|\,d\rho d\theta = \rho\sqrt{1+4\rho^2}\,d\rho d\theta. \quad \text{(1 points)}$$

Therefore,

$$\begin{aligned}\iint_S z\,dS &= \int_0^{2\pi}\int_0^2 \rho^2 \cdot \rho\sqrt{1+4\rho^2}\,d\rho d\theta \\ &= 2\pi\int_0^2 \rho^3\sqrt{1+4\rho^2}\,d\rho. \quad \text{(4 points)}\end{aligned}$$

Here,

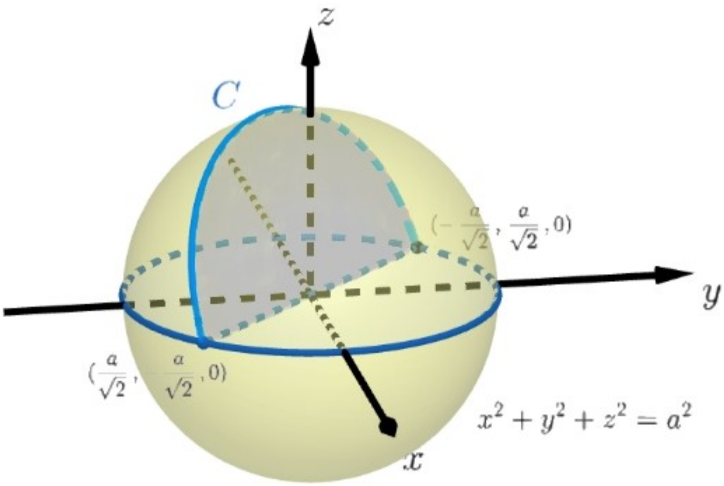
$$\begin{aligned}\int_0^2 \rho^3\sqrt{1+4\rho^2}\,dr &= \frac{\rho^2}{12}(1+4\rho^2)^{3/2}\Big|_0^2 - \frac{1}{6}\int_0^2 \rho(1+4\rho^2)^{3/2}\,d\rho \\ &= \frac{17^{3/2}}{3} - \frac{1}{120}(1+4\rho^2)^{5/2}\Big|_0^2 \\ &= \frac{17^{3/2}}{3} - \frac{17^{5/2}}{120} + \frac{1}{120}. \quad \text{(4 points; allowing small mistake)}\end{aligned}$$

All together, one obtains $\iint_S z\,dS = 2\pi\Big[\frac{17^{3/2}}{3} - \frac{17^{5/2}}{120} + \frac{1}{120}\Big]$.

3. Let $\widetilde{C}$ be the circle which is the intersection of $x^2+y^2+z^2=a^2$, $a>0$, and the plane $2x+2y+z=0$. Let C be the upper part of $\widetilde{C}$ that is above the xy-plane from $(-\frac{a}{\sqrt{2}},\frac{a}{\sqrt{2}},0)$ to $(\frac{a}{\sqrt{2}},-\frac{a}{\sqrt{2}},0)$. Consider $\mathbf{F}(x,y,z)=\left\langle x^3+y+1,\sin(y^3)+z,x+e^{z^3}\right\rangle$.

(a) (5 pts) Find curl $\mathbf{F}$

(b) (15 pts) Evaluate $\int_C \mathbf{F}\cdot d\mathbf{r}$



Solution:

(a) curl $\mathbf{F}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ \partial x & \partial y & \partial z\\ x^3+y+1 & \sin(y^3)+z & x+e^{z^3}\end{vmatrix}=\langle -1,-1,-1\rangle$

定義正確 2分, 再看每個分量, 各 1分。若沒寫定義, 直接寫答案, 每個分量 2分, 全對 5分。

(b) $\int_C \mathbf{F}\cdot d\mathbf{r}$, where $C:\ x^2+y^2+z^2=a^2,\ 2x+2y+z=0$ and $z\geq 0$.

方法 1. 硬算 line integral.

① C 的參數式 7分(檢查是否 $5x^2+5y^2+8xy=a^2$, $z=-2x-2y$)

② $\int_C \mathbf{F}\cdot d\mathbf{r}=-\frac{5}{6}\pi a^2+\sqrt{2}a$, 8分。不必檢查計算過程, 直接看答案。正負號錯扣 1分, 其他錯誤, 就是 0分。

方法 2. 利用 Stokes' Theorem

① 將 C 再加上 1條 curve L (不必是下方建議解法中的直線段) 構成一個 surface D (不一定是下方的半圓盤) 的 closed boundary, 正確寫下 Stokes' Thm 的公式, 不管接下來計算正確與否, 給 7分。 小錯, 如 curve L 的方向, orientation 等, 扣 1分。

② D 上的 surface integral, 先看 $d\mathbf{S}=\mathbf{n}dS$, $\mathbf{n}$ 正確給3分, 方向錯只給 2分。計算過程不看, 答案對不管正負號給 1分。此部分共 4分。

③ L 上的 line integral. 只檢查最後答案, 正確給 4分。視個別情形小錯扣 1分。

①②③ 獨立計分。即使①中的 L 方向錯了, ③ 中只檢查在這個 L 上的 line integral 的數值。

利用 Stokes; Theorem 的參考解法

① Let L be the line segment from $(\frac{a}{\sqrt{2}},-\frac{a}{\sqrt{2}},0)$ to $(-\frac{a}{\sqrt{2}},\frac{a}{\sqrt{2}},0)$, and let D be the semi-disc oriented such that $\partial D=C\cup L$

Stoke's Theorem implies

$$\int_{C\cup L}\mathbf{F}\cdot d\mathbf{r}=\iint_D \text{curl }\mathbf{F}\cdot d\mathbf{S}\quad (\text{正確給 7分})$$

② D is oriented with $\mathbf{n}=\left\langle\frac{2}{3},\frac{2}{3},\frac{1}{3}\right\rangle$ 正確給 3分

$$\iint_D \text{curl }\mathbf{F}\cdot d\mathbf{S}=\iint_D\langle 1,-1,-1\rangle\cdot\left\langle\frac{2}{3},\frac{2}{3},\frac{1}{3}\right\rangle dS=-\frac{5}{3}\text{Area}(D)=-\frac{5}{3}\left(\frac{\pi}{2}a^2\right)$$

$$=-\frac{5}{6}\pi a^2\quad \text{正確給 1分,}\ \text{②部份共 4分}$$

③ Let $\mathbf{r}(t) = \left\langle -\frac{a}{\sqrt{2}}t, \frac{a}{\sqrt{2}}t, 0 \right\rangle$, $t \in [-1, 1]$ be a parametrization of L. Then

$$\int_L \mathbf{F} \cdot d\mathbf{r} = \int_{-1}^{1} \left\langle \left(-\frac{a}{\sqrt{2}}t\right)^3 + \left(\frac{a}{\sqrt{2}}t\right) + 1, \sin\left(\left(\frac{a}{\sqrt{2}}t\right)^3\right), -\frac{a}{\sqrt{2}}t + 1 \right\rangle \cdot \left\langle -\frac{a}{\sqrt{2}}, \frac{a}{\sqrt{2}}, 0 \right\rangle dt$$

$$= \int_{-1}^{1} -\frac{a}{\sqrt{2}} dt + 0 \text{ Since the integral of an odd function on } [-1,1] \text{ is } 0$$

$$= -\sqrt{2}a.$$ 正確給 4分

$\therefore \int_C \mathbf{F} \cdot d\mathbf{r} = -\frac{5}{6}\pi a^2 + \sqrt{2}a$

4. Consider the vector field

$$\boldsymbol{F}(x,y,z)=\left(\frac{x}{x^2+y^2},\frac{y}{x^2+y^2},1\right).$$

Surface $S_1=\left\{(x,y,z):\frac{x^2}{9}+\frac{y^2}{4}\le 1,\ x^2+y^2\ge 1,\ z=1\right\}$ is a plane region with $\boldsymbol{n}=(0,0,1)$.

Surface $S_2=\left\{(x,y,z):\frac{x^2}{9}+\frac{y^2}{4}\le 1,\ x^2+y^2\ge 1,\ z=0\right\}$ is a plane region with $\boldsymbol{n}=(0,0,-1)$.

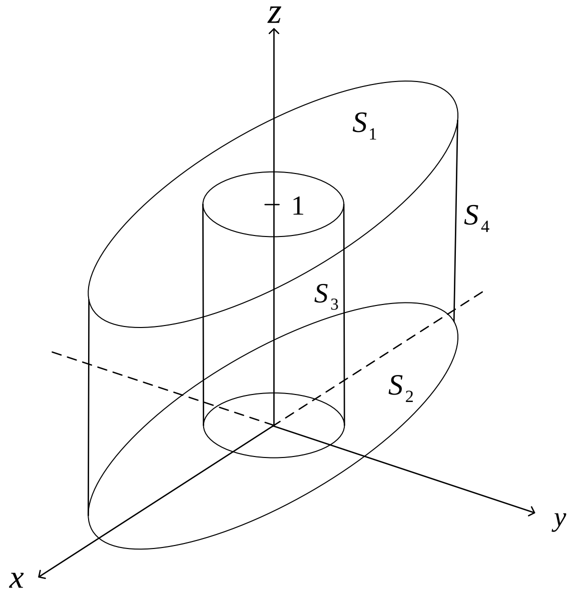
Surface $S_3=\left\{(x,y,z):x^2+y^2=1,\ 0\le z\le 1\right\}$ is a cylinder with $\boldsymbol{n}$ pointing away from the z-axis.

Surface $S_4=\left\{(x,y,z):\frac{x^2}{9}+\frac{y^2}{4}=1,\ 0\le z\le 1\right\}$ has $\boldsymbol{n}$ pointing away from the z-axis.

(a) (4 pts) Compute the flux of $\boldsymbol{F}$ across S_1 and S_2, respectively.

(b) (6 pts) Compute the flux of $\boldsymbol{F}$ across S_3.

(c) (6 pts) Compute the flux of $\boldsymbol{F}$ across S_4.



Solution:

(a) The flux of $\boldsymbol{F}$ across S_1 is

$$\iint_{S_1}\boldsymbol{F}\cdot\boldsymbol{n}\,dS=\iint_{S_1}\left(\frac{x}{x^2+y^2},\frac{y}{x^2+y^2},1\right)\cdot(0,0,1)\,dS=\iint_{S_1}dS \qquad \text{(1 point)}$$

$$=\text{Area}(S_1)=6\pi-\pi=5\pi. \qquad \text{(1 point)}$$

The flux of $\boldsymbol{F}$ across S_2 is

$$\iint_{S_2}\boldsymbol{F}\cdot\boldsymbol{n}\,dS=\iint_{S_2}\left(\frac{x}{x^2+y^2},\frac{y}{x^2+y^2},1\right)\cdot(0,0,-1)\,dS=-\iint_{S_2}dS \qquad \text{(1 point)}$$

$$=-\text{Area}(S_2)=-5\pi. \qquad \text{(1 point)}$$

(b) **Method 1**: Note that the outward normal of S_3 is given by

$$\boldsymbol{n}=(x,y,0). \qquad \text{(2 points)}$$

So the flux of $\boldsymbol{F}$ across S_3 is

$$\iint_{S_3}\boldsymbol{F}\cdot\boldsymbol{n}\,dS=\iint_{S_3}\left(\frac{x}{x^2+y^2},\frac{y}{x^2+y^2},1\right)\cdot(x,y,0)\,dS=\iint_{S_3}dS \qquad \text{(2 points)}$$

$$= \text{Area}\,(S_3) = 2\pi. \qquad \text{(2 points)}$$

Method 2: The surface S_3 can be parametrized as

$$(x, y, z) = (\cos\theta, \sin\theta, z), \qquad 0 \le \theta \le 2\pi,\ 0 \le z \le 1, \qquad \text{(1 point)},$$

so

$$\boldsymbol{n}\,dS = (-\sin\theta, \cos\theta, 0) \times (0, 0, 1)\,d\theta\,dz = (\cos\theta, \sin\theta, 0)\,d\theta\,dz. \qquad \text{(2 points)}$$

Then the flux of $\boldsymbol{F}$ across S_3 is

$$\iint_{S_3} \boldsymbol{F} \cdot \boldsymbol{n}\,dS = \int_0^1 \int_0^{2\pi} (\cos\theta, \sin\theta, 1) \cdot (\cos\theta, \sin\theta, 0)\,d\theta\,dz \qquad \text{(2 points)}$$

$$= \int_0^1 \int_0^{2\pi} d\theta\,dz = 2\pi. \qquad \text{(1 point)}$$

(c) Let E be the solid region enclosed by the surfaces S_1, S_2, S_3, and S_4. Using the divergence theorem we get

$$\iiint_E \nabla \cdot \boldsymbol{F}\,dV =$$

$$\iint_{S_1} \boldsymbol{F} \cdot \boldsymbol{n}\,dS + \iint_{S_2} \boldsymbol{F} \cdot \boldsymbol{n}\,dS - \iint_{S_3} \boldsymbol{F} \cdot \boldsymbol{n}\,dS + \iint_{S_4} \boldsymbol{F} \cdot \boldsymbol{n}\,dS \qquad \text{(3 points)}$$

Since

$$\nabla \cdot \boldsymbol{F} = \frac{\partial}{\partial x}\left(\frac{x}{x^2 + y^2}\right) + \frac{\partial}{\partial y}\left(\frac{y}{x^2 + y^2}\right)$$

$$= \frac{\left(x^2 + y^2\right) - x \cdot 2x}{\left(x^2 + y^2\right)^2} + \frac{\left(x^2 + y^2\right) - y \cdot 2y}{\left(x^2 + y^2\right)^2} = 0 \qquad \text{in } E, \qquad \text{(2 points)}$$

we obtain

$$5\pi - 5\pi - 2\pi + \iint_{S_4} \boldsymbol{F} \cdot \boldsymbol{n}\,dS = 0$$

$$\Rightarrow \iint_{S_4} \boldsymbol{F} \cdot \boldsymbol{n}\,dS = 2\pi. \qquad \text{(1 point)}$$

5. (a) (8 pts) Determine whether $\sum_{n=1}^{\infty}\left(\frac{2n}{2n+1}-\frac{2n-1}{2n}\right)$ is convergent or divergent. Compute the sum if it converges.

(b) (12 pts) Find the interval of convergence of the power series $\sum_{n=3}^{\infty}\left(\frac{\ln(n+1)-\ln n}{\ln(n)}\right)x^n$.

Solution:

(a) 此題的最後級數數值為$1-\ln 2$。這部分佔1分。其餘的7分配分比例如下:

(7分) 有正確地說明級數收斂。例如採用下列兩個方法:

- 觀察到$0\leqslant\frac{2n}{2n+1}-\frac{2n-1}{2n}=\frac{1}{(2n)(2n+1)}\leqslant\frac{1}{4n^2}$並使用 comparison test。
- (以下將原級數的前n項部份和記為s_n。)正確說明交錯級數$\frac{1}{2}-\frac{1}{3}+\frac{1}{4}+\cdots$收斂，並(以某種形式考慮)$s_n$與上述交錯級數的比較，藉此說明原級數也收斂(到同一個極限)。如果只從交錯級數收斂就做出原級數收斂的結論不能算對(參看下一條)。

(4分) 正確說明交錯級數$\frac{1}{2}-\frac{1}{3}+\frac{1}{4}+\cdots$收斂，但直接宣稱原級數收斂。

(1-3分) The answer might conclude divergence by applying tests with correct reasons to wrongly obtained expressions. If this is the case, the grader may give 1 to 3 points depending the situations.

(b) Let $a_n=\frac{\ln(n+1)-\ln n}{\ln n}$. One see that $\lim\frac{a_n}{a_{n+1}}=1$ and hence the radius of convergence is 1. On the other hand, by the mean value theorem we have $\ln(n+1)-\ln n>\frac{1}{n+1}$, and hence

$$\frac{\ln(n+1)-\ln n}{\ln n}\geqslant\frac{1}{(n+1)\ln n}.$$

Since $\frac{1}{(x+1)\ln x}$ is decreasing and nonnegative and $\int_3^\infty\frac{1}{(x+1)\ln x}dx>\int_3^\infty\frac{1}{(x+1)\ln(x+1)}dx$ diverges, we see that $x=1$ does not lie in the interval of convergence. As for $x=-1$, the series

$$\sum_{n=3}^{\infty}(-1)^n\frac{\ln(n+1)-\ln n}{\ln n}=\sum_{n=3}^{\infty}(-1)^n\left(\frac{\ln(n+1)}{\ln n}-1\right)$$

is alternating. We examine if its terms get smaller and smaller as $n\nearrow$: the function $\frac{\ln(x+1)}{\ln x}-1$ has derivative

$$\frac{x\ln x-(x+1)\ln(x+1)}{x(x+1)(\ln x)^2}<0.$$

Finally,

$$\frac{\ln(n+1)-\ln n}{\ln n}=\frac{\ln(n+1)}{\ln n}-1\searrow 0$$

as can be seen by L'Hospital. According Leibniz's test, -1 lies in the interval of convergence.

此題滿分12分，分成兩部分:

(i) 收斂半徑(5分):

(5分) 正確求得收斂半徑。以下是兩種可接受的寫法:

- 將冪級數係數記為a_n，並正確求得

$$\lim_{n\longrightarrow\infty}\left|\frac{a_n}{a_{n+1}}\right|=1\quad\text{或}\quad\lim_{n\longrightarrow\infty}\frac{1}{|a_n|^{\frac{1}{n}}}=1。$$

- 令冪級數的第n項為b_n並正確算出

$$\lim_{n\longrightarrow\infty}\left|\frac{b_{n+1}}{b_n}\right|=|x|\quad\text{或}\quad\lim_{n\longrightarrow\infty}|b_n|^{\frac{1}{n}}=|x|,$$

接著「直接」宣稱由ratio test或root test可知收斂半徑為1，或更詳細地說明$|x|<1$時級數收斂且$|x|>1$時級數發散。這裡的重點是如果學生只寫其中一邊(多半是$|x|<1$這邊)是不夠的。如果只是考慮$x=1$的情況就說收斂半徑是1也是不夠的。參看下一條。

(1分) 正確說明冪級數在某端點收斂，但直接宣稱這保證收斂半徑是1。此時在收斂半徑的部分只給1分。(但是他會在端點處理的部分的分，見下面(ii)項。)

(1-3分) 如果是想採用收斂半徑公式、ratio test或root test來求得半徑，但是極限的數值計算錯誤，可以給1至3分，由助教自行判斷。

(ii)端點處理(7分): 右端點$x=1$佔4分、左端點$x=-1$佔3分，有做出適當的嘗試可得部分分，由助教自行判斷。

6. Let $f(x) = \int_0^x \sqrt{4+t^3}\, dt$.

(a) (6 pts) Write down the Maclaurin series (the Taylor series at 0) for $\sqrt{4+t^3}$ and find its radius of convergence.

(b) (6 pts) Write down the Maclaurin series for $f(x)$ and find its radius of convergence.

(c) (8 pts) Write down $T_4(x)$, the fourth degree Taylor polynomial of $f(x)$ at 0.
Estimate $\left|f(\frac{1}{2}) - T_4(\frac{1}{2})\right|$. (You need to justify your answer.)

Solution:

(a)

$$\sqrt{4+t^3} = 2\sqrt{1+\frac{t^3}{4}} = 2\left(\sum_{n=0}^{\infty} C_n^{\frac{1}{2}} \left(\frac{t^3}{4}\right)^n\right) = \sum_{n=0}^{\infty} C_n^{\frac{1}{2}} \cdot \frac{1}{2^{2n-1}} t^{3n}. \ (3pts)$$

The radius of convergence of the binomial series is 1. Thus the series converges absolutely if $\left|\frac{t^3}{4}\right| < 1$ and it diverges if $\left|\frac{t^3}{4}\right| > 1$ (2 pts). Hence the radius of convergence of the series is $\sqrt[3]{4} = 2^{\frac{2}{3}}$. (1 pt)
(Or by the Ratio Test,

$$\lim_{n\to\infty} \left|\frac{C_{n+1}^{\frac{1}{2}} \cdot \frac{1}{2^{2n+1}} t^{3n+3}}{C_n^{\frac{1}{2}} \cdot \frac{1}{2^{2n-1}} t^{3n}}\right| = \lim_{n\to\infty} \frac{n-\frac{1}{2}}{n+1} \frac{|t|^3}{4} = \frac{|t|^3}{4}. \ (2pts)$$

Hence, the Maclaurin series converges if $\left|\frac{t^3}{4}\right| < 1$ and the series diverges if $\left|\frac{t^3}{4}\right| > 1$. Hence the radius of convergence is $\sqrt[3]{4} = 2^{\frac{2}{3}}$. (1 pt))

(b)

$$f(x) = \int_0^x \sqrt{4+t^3} dt = \int_0^x \left(\sum_{n=0}^{\infty} C_n^{\frac{1}{2}} \cdot \frac{1}{2^{2n-1}} t^{3n}\right) dt = \sum_{n=0}^{\infty} C_n^{\frac{1}{2}} \frac{1}{2^{2n-1}} \cdot \frac{1}{3n+1} x^{3n+1}. (3pts)$$

The radius of convergence is still $2^{\frac{2}{3}}$ because a power series and its integral have the same radius of convergence. (3 pts)
(Or you can apply the Ratio Test,

$$\lim_{n\to\infty} \left|\frac{C_{n+1}^{\frac{1}{2}} \cdot \frac{1}{2^{2n+1}} \frac{1}{3n+4} x^{3n+4}}{C_n^{\frac{1}{2}} \cdot \frac{1}{2^{2n-1}} \frac{1}{3n+1} x^{3n+1}}\right| = \lim_{n\to\infty} \frac{n-\frac{1}{2}}{n+1} \frac{3n+1}{3n+4} \frac{|x|^3}{4} = \frac{|x|^3}{4}. \ (2pts)$$

Hence, the series converges if $\left|\frac{x^3}{4}\right| < 1$ and the series diverges if $\left|\frac{x^3}{4}\right| > 1$. Hence the radius of convergence is $\sqrt[3]{4} = 2^{\frac{2}{3}}$. (1 pt))

(c) $T_4(x) = 2x + \frac{1}{16}x^4$. (2 pts)

$f(\frac{1}{2}) = 2 \cdot \frac{1}{2} + \sum_{n=1}^{\infty} C_n^{\frac{1}{2}} \frac{1}{2^{2n-1}} \frac{1}{3n+1} \frac{1}{2^{3n+1}} = 1 + \sum_{n=1}^{\infty} C_n^{\frac{1}{2}} \frac{1}{2^{5n}} \frac{1}{3n+1}$ and the series $\sum_{n=1}^{\infty} C_n^{\frac{1}{2}} \frac{1}{2^{5n}} \frac{1}{3n+1}$ is an alternating series (2 pts).

Moreover, $|C_n^{\frac{1}{2}} \frac{1}{2^{5n}} \frac{1}{3n+1}|$ is decreasing and approaches 0 as $n \to \infty$

(If students CLAIM / MENTION that the sequence is decreasing and tends to 0, then they get 2 points. The complete argument is as below.
The sequence $|C_n^{\frac{1}{2}} \frac{1}{2^{5n}} \frac{1}{3n+1}|$ is decreasing because

$$\frac{|C_{n+1}^{\frac{1}{2}} \frac{1}{2^{5n+5}} \frac{1}{3n+4}|}{|C_n^{\frac{1}{2}} \frac{1}{2^{5n}} \frac{1}{3n+1}|} = \frac{n-\frac{1}{2}}{n+1} \frac{1}{2^5} \frac{3n+1}{3n+4} < \frac{1}{2^5} < 1 \text{ for } n \geq 1.$$

Also by the above ratio, we know that $|C_n^{\frac{1}{2}} \frac{1}{2^{5n}} \frac{1}{3n+1}| \leq \frac{1}{2^{5n}}$ for $n \geq 1$. Thus it approaches 0 as $n \to \infty$.)

Hence by the Alternating Series Estimation Theorem,

$$|f(\frac{1}{2}) - T_4(\frac{1}{2})| = \left|\sum_{n=2}^{\infty} C_n^{\frac{1}{2}} \frac{1}{2^{5n}} \frac{1}{3n+1}\right| \leq |C_2^{\frac{1}{2}} \frac{1}{2^{10}} \frac{1}{6+1}| = \frac{1}{2^{13}7} \quad (2pts)$$

Students may use Taylor's inequality to estimate the error. Writing down the inequality $|f(\frac{1}{2}) - T_4(\frac{1}{2})| \leq \frac{M}{5!}(\frac{1}{2})^5$, where M is an upper bound for $|f^{(5)}(x)|$ on the interval $[0, \frac{1}{2}]$, students can get 2 points.

It is complicated to estimate $|f^{(5)}(x)|$. If students have some progress in computing $|f^{(5)}(x)|$, they can get 1 more point.