1. (10 pts) Let the curve C be given by $\mathbf{r}(t) = \frac{2}{3}(2+t)^{\frac{3}{2}}\,\mathbf{i} + \frac{2}{3}(2-t)^{\frac{3}{2}}\,\mathbf{j} + at\,\mathbf{k}$, $a \neq 0$, $t \in (-2,2)$. Find the vectors $\mathbf{T}$, $\mathbf{N}$, $\mathbf{B}$, the curvature κ and the torsion τ of the curve C at $t = 0$.

Solution:

$$\begin{aligned}
\mathbf{r}'(t) &= \left\langle (2+t)^{1/2}, -(2-t)^{1/2}, a \right\rangle \\
\mathbf{r}''(t) &= \frac{1}{2}\left\langle (2+t)^{-1/2}, (2-t)^{-1/2}, 0 \right\rangle \\
\mathbf{r}'''(t) &= \frac{1}{4}\left\langle -(2+t)^{-3/2}, (2-t)^{-3/2}, 0 \right\rangle
\end{aligned}$$

$$\begin{aligned}
|\mathbf{r}'(t)| &= (4+a^2)^{1/2} \\
\mathbf{r}'(t) \times \mathbf{r}''(t) &= \frac{1}{2}\left\langle -a(2-t)^{-1/2}, a(2+t)^{-1/2}, 4(4-t^2)^{-1/2} \right\rangle \\
|\mathbf{r}'(t) \times \mathbf{r}''(t)| &= \sqrt{4+a^2}/\sqrt{4-t^2},\ (\mathbf{r}'(t) \times \mathbf{r}''(t)) \cdot \mathbf{r}'''(t) = \frac{a}{2}(4-t^2)^{-3/2}
\end{aligned}$$

$$\begin{aligned}
\mathbf{T}(t) &= \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \frac{1}{\sqrt{4+a^2}}\left\langle (2+t)^{1/2}, -(2-t)^{1/2}, a \right\rangle \\
\mathbf{N}(t) &= \frac{\mathbf{T}'}{|\mathbf{T}'|} = \frac{1}{2}\left\langle (2-t)^{1/2}, (2+t)^{1/2}, 0 \right\rangle \\
\mathbf{B}(t) &= \mathbf{T} \times \mathbf{N}(t) = \frac{1}{2\sqrt{4+a^2}}\left\langle -a(2+t)^{1/2}, a(2-t)^{1/2}, 4 \right\rangle \\
\kappa(t) &= \frac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|} = \frac{|\mathbf{r}' \times \mathbf{r}''|}{|\mathbf{r}'|^3} = \frac{1}{(4+a^2)\sqrt{4-t^2}} \\
\tau(t) &= \frac{(\mathbf{r}' \times \mathbf{r}'') \cdot \mathbf{r}'''}{|\mathbf{r}' \times \mathbf{r}''|^2} = \frac{a}{2}\frac{1}{(4+a^2)\sqrt{4-t^2}}
\end{aligned}$$

At $t = 0$ (每項 2 分)

$$\begin{aligned}
\mathbf{T} &= \frac{1}{\sqrt{4+a^2}}\left\langle \sqrt{2}, -\sqrt{2}, a \right\rangle \\
\mathbf{N} &= \frac{1}{2}\langle 2, 2, 0 \rangle \\
\mathbf{B} &= \frac{1}{2\sqrt{4+a^2}}\left\langle -a\sqrt{2}, a\sqrt{2}, 4 \right\rangle \\
\kappa &= \frac{1}{2(4+a^2)} \\
\tau &= \frac{a}{4(4+a^2)}
\end{aligned}$$

2. (12 pts) Consider the following function on $\mathbf{R}^2$:

$$f(x,y) = \begin{cases} \frac{x^2 y}{x^4+(\sin y)^2} & \text{if} \quad (x,y) \neq (0,0); \\ 0 & \text{if} \quad (x,y) = (0,0). \end{cases}$$

(a) (6 pts) Is $f(x,y)$ continuous at $(x,y) = (0,0)$? Justify your answer.

(b) (4 pts) Do the partial derivatives $f_x(0,0)$ and $f_y(0,0)$ exist? (Compute them if you think they exist; otherwise, prove that they do not exist.)

(c) (2 pts) Is f differentiable at $(0,0)$? Justify your answer.

Solution:

(a) Let (x,y) tend to $(0,0)$ along special paths. One may gain 6 points if the students find such a path that along which the limit of f is different from $0 = f(0,0)$. For example, one may consider (x, mx^2): if $x \neq 0$,

$$f(x,mx) = \frac{x^2 \cdot mx^2}{x^4 + \sin(mx)^2} = \frac{m}{1+(\frac{m\sin(mx^2)}{mx^2})^2} \to \frac{m}{1+m^2} \neq 0 = f(0,0) \quad \text{if} \quad m \neq 0 \quad \text{as} \quad x \to 0.$$

Therefore f is not continuous at $(0,0)$. One may also consider for example the curve $y = \sin^{-1}(mx^2)$. 找尋路徑並說明上述極限不等於$f(0,0) = 0$的任何一個環節錯誤可扣1至2分。

(b) Compute by definition that $f_x(0,0) = 0 = f_y(0,0)$. 過程與計算結果均正確者得4分，否則得0分。

(c) Since f is not continuous at $(0,0)$, it is not differentiable at $(0,0)$. One may also argue by definition. 理由充分者得2分，否則得0分。

3. (10 pts) Let $g(x,y,z)$ be a function defined on $\mathbb{R}^3$ with continuous partial derivatives. Suppose that

$$|\nabla g(2,1,3)|^2 = 24 \text{ and } g_z(2,1,3) > 0.$$

Moreover, the trajectories of the two curves

$$\mathbf{r}_1(s) = \langle 2s, s^2, 1+2s \rangle \quad \text{and} \quad \mathbf{r}_2(t) = \langle 2e^t, \cos t, 3+t+5t^2 \rangle$$

lie on the level surface $g(x,y,z) = 0$ completely.

(a) (5 pts) Find the vector $\nabla g(2,1,3)$.

(b) (5 pts) Suppose that $f(x,y,z)$ is a function defined on $\mathbb{R}^3$ with continuous partial derivatives such that

$$f(2,1,3) \geqslant f(x,y,z) \text{ for every point } (x,y,z) \text{ on the level surface } g(x,y,z) = 0.$$

If $f(2,1,3) = 5$, $|\nabla f(2,1,3)|^2 = 6$ and $f_y(2,1,3) > 0$, estimate the value of $f(2.01, 0.9, 3.02)$ by the linear approximation of f at $(2,1,3)$.

Solution:

(a) Note that $\mathbf{r}_1(1) = (2,1,3) = \mathbf{r}_2(0)$. Thus

$$(2,2,2) = \mathbf{r}_1'(1) \perp \nabla g(2,1,3) \perp \mathbf{r}_2'(0) = (2,0,1),$$

（到這裡兩個切向量都計算正確可得1分） and hence $\nabla g(2,1,3)$ is parallel to $(2,2,2) \times (2,0,1) = (2,2,-4)$. （指出$\nabla g(2,1,3)$平行於兩切向量外積可得2分；外積計算正確得1分） By (i), we see that $\nabla g(2,1,3) = (-2,-2,4)$. （用到條件(i)來決定$\nabla g(2,1,3)$的方向可得1分）

(b) We need to find $\nabla f(2,1,3)$ for the linear approximation. By (a) and by the Lagrange multiplier method we see that $\nabla f(2,1,3) = \lambda \nabla g(2,1,3)$ for some $\lambda \in \mathbf{R}$.（說到要使用Lagrange 得1分） By (b) we see that $\lambda = \frac{-1}{2}$ and $\nabla f(2,1,3) = (1,1,-2)$.（決定$\nabla f(2,1,3) = (1,1,-2)$的理由正確可得2分） Therefore

$$f(2.01, 0.9, 3.02) \approx f(2,1,3) + \nabla f(2,1,3) \cdot (0.01, -0.1, 0.02) = 4.87.$$

（線性逼近的形式正確得1分，計算正確得1分）

4. (10 pts) Let $f(x,y) = \frac{xy(x+y)}{e^{x+y}}$ be defined on the first quadrant $D : x > 0$ and $y > 0$ (without boundary). Find all critical points of f in D and classify them (as local maximum points, local minimum points, or saddle points). Please provide details of calculation.

Solution:

The first derivatives of f are

$$\frac{\partial f}{\partial x} = e^{-x-y}\left(2xy + y^2 - x^2y - xy^2\right),$$

$$\frac{\partial f}{\partial y} = e^{-x-y}\left(2xy + x^2 - x^2y - xy^2\right). \qquad \textbf{(2 points)}$$

There is only one critical point of f in D, which is

$$(x,y) = \left(\frac{3}{2}, \frac{3}{2}\right). \qquad \textbf{(2 points)}$$

The second partial derivatives of f are

$$\frac{\partial^2 f}{\partial x^2} = e^{-x-y}\left(2y - 4xy - 2y^2 + x^2y + xy^2\right),$$

$$\frac{\partial^2 f}{\partial x \partial y} = e^{-x-y}\left(2x + 2y - x^2 - 4xy - y^2 + x^2y + xy^2\right),$$

$$\frac{\partial^2 f}{\partial y^2} = e^{-x-y}\left(2x - 4xy - 2x^2 + x^2y + xy^2\right). \qquad \textbf{(3 points)}$$

Since

$$\det \nabla^2 f\left(\frac{3}{2}, \frac{3}{2}\right) = e^{-6}\begin{vmatrix} -\frac{15}{4} & -\frac{3}{4} \\ -\frac{3}{4} & -\frac{15}{4} \end{vmatrix} > 0$$

and

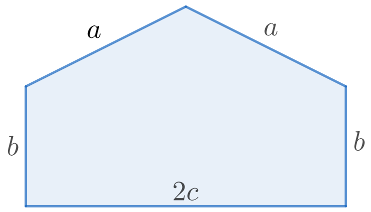
$$\frac{\partial^2 f}{\partial x^2}\left(\frac{3}{2}, \frac{3}{2}\right) = -\frac{15}{4}e^{-3} < 0 \qquad \textbf{(2 points)},$$

the critical point $\left(\frac{3}{2}, \frac{3}{2}\right)$ is a local maximum point of f (**1 point**).

5. (12 pts) A pentagon is formed by placing an isosceles triangle on a rectangle. The side lengths are denoted by a, b, and c as shown in the figure.

(a) (3 pts) Write down the area of pentagon in terms of a, b, and c.

(b) (9 pts) Find the maximum area of pentagon if the perimeter is fixed as 2.



Solution:

(a) The height of the upper triangle equals $\sqrt{a^2 - c^2}$ `(2 points)`. Therefore the area is given by

$$A = c \cdot \sqrt{a^2 - c^2} + 2bc \quad \texttt{(2 points)}.$$

(b) We use the method of Lagrange multiplier. We are looking for the maximum value of A under the conditions $a, b, c > 0$ and $g(a, b, c) = 1$ where $g(a, b, c) = a + b + c$. When A achieves the extremum, we have

$$\begin{aligned} \frac{ac}{\sqrt{a^2 - c^2}} &= \lambda \\ 2c &= \lambda \\ \frac{a^2 - 2c^2}{\sqrt{a^2 - c^2}} + 2b &= \lambda \\ a + b + c &= 1 \end{aligned} \qquad \texttt{(4 points)}.$$

The first two equations imply $3a^2 = 4c^2$ or equivalently $c = \frac{\sqrt{3}}{2}a$. Plugging into the third equation and replacing λ by $2c = \sqrt{3}a$, we have $b = \frac{1+\sqrt{3}}{2}a$. The last equation then reduces to $\frac{3+2\sqrt{3}}{2}a = 1$ so we obtain

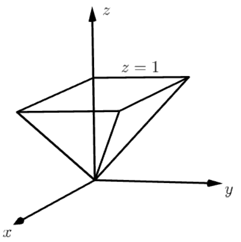
$$a = \frac{2}{3 + 2\sqrt{3}}, \quad b = \frac{1 + \sqrt{3}}{3 + 2\sqrt{3}}, \quad c = \frac{\sqrt{3}}{3 + 2\sqrt{3}}.$$

In this circumstance,

$$A = \frac{6 + 3\sqrt{3}}{(3 + 2\sqrt{3})^2} \quad \texttt{(4 points)}.$$

6. (26 pts) (a) (6 pts) Find the average value of $f(x) = \int_x^a e^{-t^2} dt$ on the interval $[0, a]$, where $a > 0$ is a constant.

(b) (10 pts) Compute $\iiint_E e^{3y-y^3} dV$, where E is the solid bounded by $x = 0$, $y = 0$, $x = z$, $y = z$, and $z = 1$.



(c) (10 pts) Compute $\int_{-a}^{a} \int_0^{\sqrt{a^2-x^2}} \int_a^{a+\sqrt{a^2-x^2-y^2}} \frac{1}{\sqrt{x^2+y^2+z^2}} dzdydx.$

(Hint: Use Spherical coordinates.)

Solution:

(a) The average value of $f(x)$ on $[0, a]$ is $\frac{1}{a}\int_0^a f(x)\,dx$. (1 pt for the definition of average value.)

$$\begin{aligned}\frac{1}{a}\int_0^a f(x)\,dx &= \frac{1}{a}\int_0^a \int_x^a e^{-t^2}\,dt\,dx \\ &= \frac{1}{a}\int_0^a \int_0^t e^{-t^2}\,dx\,dt \quad \text{(3 pts for changing the order of integration)} \\ &= \frac{1}{a}\int_0^a te^{-t^2}\,dt = \frac{1}{a}\left(-\frac{1}{2}e^{-t^2}\right)\Big|_{t=0}^{t=a} = \frac{1}{2a}(1-e^{-a^2}) \text{ (2 pts for the final answer)}\end{aligned}$$

(b) **Solution 1:** $E = \{(x, y, z)|0 \le y \le 1, y \le z \le 1, 0 \le x \le z\}$

Hence $\iiint_E e^{3y-y^3}\,dV = \int_0^1 \int_y^1 \int_0^z e^{3y-y^3}\,dxdzdy$

(5 pts. If students correctly project E onto the yz-plane and write down the correct range of x, they get 2 pts. 3 pts for correct iterated integrals.)

$$\begin{aligned}\int_0^1 \int_y^1 \int_0^z e^{3y-y^3}\,dxdzdy &= \int_0^1 \int_y^1 ze^{3y-y^3}\,dz \quad \text{(1 pt)} \\ &= \int_0^1 \frac{1}{2}(1-y^2)e^{3y-y^3}\,dy \quad \text{(1 pt)} \\ &\overset{\text{let } u=3y-y^3}{\underset{du=(3-3y^2)dy}{=\!=\!=}} \int_0^2 \frac{1}{6}e^u\,du\end{aligned}$$

(1 pt for substitution. 1 pt for correct upper and lower limit.)

$$= \frac{1}{6}(e^2 - 1) \quad \text{(1 pt for final answer.)}$$

Solution 2: $E = \{(x, y, z)|0 \le y \le 1, y \le x \le 1, x \le z \le 1\} \cup \{(x, y, z)|0 \le y \le 1, 0 \le x \le y, y \le z \le 1\}$

$$\iiint_E e^{3y-y^3} dV = \int_0^1 \int_t^1 \int_x^1 e^{3y-y^3} dzdxdy + \int_0^1 \int_0^y \int_y^1 e^{3y-y^3} dzdxdy$$

(5 pts. If students correctly project E onto the xy-plane and write down the correct range of x, they get 2 pts. 3pts for correct iterated integrals.)

The integral is

$$\int_0^1\int_y^1 (1-x)e^{3y-y^3}\,dxdy + \int_0^1\int_0^y (1-y)e^{3y-y^3}\,dxdy \quad (1\text{ pt})$$
$$= \int_0^1 (1-y) - \frac{1}{2}(1-y^2)e^{3y-y^3}\,dy + \int_0^1 y(1-y)e^{3y-y^3}\,dy \quad (1\text{ pt})$$
$$= \int_0^1 \frac{1}{2}(1-y^2)e^{3y-y^3}\,dy = \frac{1}{6}(e^2-1) \quad (3\text{ pts})$$

(c) **Solution 1:** The integral is $\iiint\limits_E \frac{1}{\sqrt{x^2+y^2+z^2}}dV$, where $E = \{(\rho,\theta,\varphi)|0 \le \theta \le \pi, 0 \le \varphi \le \frac{\pi}{4}, \frac{a}{\cos\varphi} \le \rho \le 2a\cos\varphi\}$.

$$\begin{pmatrix} \text{1 pt for the range of } \theta \\ \text{2 pts for the range of } \varphi \\ \text{2 pts for the range of } \rho \end{pmatrix}$$

$$\iiint\limits_E \frac{1}{\sqrt{x^2+y^2+z^2}} = \int_0^\pi \int_0^{\frac{\pi}{4}} \int_{\frac{a}{\cos\varphi}}^{2a\cos\varphi} \frac{1}{\rho}\rho^2 \sin\varphi\, d\rho d\varphi d\theta \quad (1\text{ pt for Jacobian})$$
$$= \pi \int_0^{\frac{\pi}{4}} \frac{1}{2}\left(4a^2\cos^2\varphi - \frac{a^2}{\cos^2\varphi}\right)\sin\varphi\, d\varphi \quad (1\text{ pt})$$
$$= \frac{\pi}{2}a^2 \int_0^{\frac{\pi}{4}} \left(4\cos^2\varphi - \frac{1}{\cos^2\varphi}\right)\sin\varphi\, d\varphi$$
$$\xlongequal[du=-\sin\varphi\, d\varphi]{u=\cos\varphi} \frac{\pi}{2}a^2 \int_1^{\frac{1}{\sqrt{2}}} \left(4u^2 - \frac{1}{u^2}\right)(-du) \quad (2\text{ pts for substitution})$$
$$= \frac{\pi}{2}a^2 \left[\frac{4}{3}u^3 + \frac{1}{u}\right]\Bigg|_{u=\frac{1}{\sqrt{2}}}^{u=1}$$
$$= \frac{\pi}{2}a^2\left[\frac{7}{3} - \frac{4}{3}\sqrt{2}\right] \quad (1\text{ pt for final answer.})$$

Solution 2: Use cylindrical coordinates. The integral is $\iiint\limits_E \frac{1}{\sqrt{x^2+y^2+z^2}}dV$, where $E = \{(r,\theta,z)|0 \le \theta \le \pi, 0 \le r \le a, a \le z \le a + \sqrt{a^2-r^2}\}$.

$$\begin{pmatrix} \text{1 pt for the range of } \theta \\ \text{1 pt for the range of } r \\ \text{1 pt for the range of } z \end{pmatrix}$$

$$\iiint\limits_E \frac{1}{\sqrt{x^2+y^2+z^2}} = \int_0^\pi \int_0^a \int_a^{a+\sqrt{a^2-r^2}} \frac{1}{\sqrt{r^2+z^2}} r\, dzdrd\theta \quad (1\text{ pt for Jacobian})$$

Note that $\int \frac{1}{\sqrt{a^2+t^2}}dt = \ln(t+\sqrt{a^2+t^2}) + c$. (2 pts)

7. (10 pts) Let D be an xy-plane region bounded by one loop of $r^2 = \cos 2\theta$. Find the area of the part of the upper half sphere $z = \sqrt{1 - x^2 - y^2}$ that is above D.

Solution:

The first derivatives of $f(x, y) = \sqrt{1 - x^2 - y^2}$ are

$$\frac{\partial f}{\partial x} = \frac{-x}{\sqrt{1 - x^2 - y^2}},$$

$$\frac{\partial f}{\partial y} = \frac{-y}{\sqrt{1 - x^2 - y^2}}. \qquad \textbf{(2 points)}$$

The area of the graph of f above D is given by

$$\iint_D \sqrt{\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 + 1}\, dx\, dy = \iint_D \frac{1}{\sqrt{1 - x^2 - y^2}}\, dx\, dy \qquad \textbf{(2 points)}$$

$$= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \int_0^{\sqrt{\cos(2\theta)}} \frac{r}{\sqrt{1 - r^2}}\, dr\, d\theta \qquad \textbf{(3 points)}$$

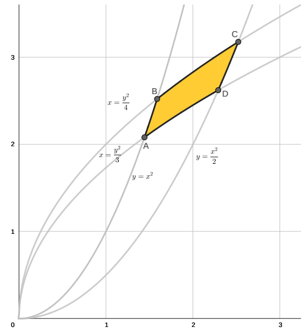
$$= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \left(1 - \sqrt{1 - \cos(2\theta)}\right) d\theta$$

$$= 2\int_0^{\frac{\pi}{4}} \left(1 - \sqrt{2}\sin\theta\right) d\theta = 2\left(\frac{\pi}{4} + 1 - \sqrt{2}\right) \qquad \textbf{(3 points)}.$$

8. (10 pts) Let $D = \{(x,y) | x > 0, y > 0, y \le x^2 \le 2y, 3x \le y^2 \le 4x\}$. Evaluate $\displaystyle\iint_D xy\,dA$.

Solution:

Method 1:



$A(3^{1/3}, 3^{2/3})$ $B(4^{1/3}, 4^{2/3})$ $C(2^{4/3}, 2^{5/3})$ $D(2^{2/3}3^{1/3}, 2^{1/3}3^{2/3})$

$$I = \int_{3^{1/3}}^{4^{1/3}} \int_{\sqrt{3x}}^{x^2} xy\,dydx + \int_{4^{1/3}}^{2^{2/3}3^{1/3}} \int_{\sqrt{3x}}^{\sqrt{4x}} xy\,dydx + \int_{2^{2/3}3^{1/3}}^{2^{4/3}} \int_{x^2/2}^{\sqrt{4x}} xy\,dydx = \frac{7}{4}$$

6 個積分上下限，每個上、下全對給 1 分，獨立給分，共 6 分。不看積分的計算，核驗最後答案 ($\frac{7}{4}$)，4 分

Method 2:

Make a change of variables $u = \frac{x^2}{y}$, $v = \frac{y^2}{x}$, which transforms D to $E = \{(u,v) | 1 \le u \le 2, 3 \le v \le 4\}$.

Observe that $uv = xy$, $\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{2x}{y} & -\frac{x^2}{y^2} \\ -\frac{y^2}{x^2} & \frac{2y}{x} \end{vmatrix} = 4 - 1 = 3$, $\frac{\partial(x,y)}{\partial(u,v)} = \frac{1}{3}$ (4 分，若沒取倒數扣 1 分)

$$I = \iint_E \underbrace{uv}_{\text{變換後的函數 2 分}} \left| \frac{\partial(x,y)}{\partial(u,v)} \right| dudv = \frac{1}{3} \underbrace{\int_1^2 u\,du \int_3^4 v\,dv}_{\text{2 組上、下限各 1 分，共 2 分}} = \frac{1}{3} \left(\frac{u^2}{2}\right)_1^2 \left(\frac{v^2}{2}\right)_3^4 = \underbrace{\frac{7}{4}}_{\text{2 分}}$$

任何有效或無效的變換，皆如下獨立個別給分：2 組上、下限各 1 分，共 2 分；Jacobian 4 分；變換後的被積分函數 2 分；答案 2 分。(計算過程不必看)