1. (15 pts) Let $f(x,y,z) = \sin(xy+z)$, and P be the point $(0,-2,\frac{\pi}{3})$.
   (a) (6 pts) Compute $\nabla f(x,y,z)$.
   (b) (2 pts) At P, find the direction along which f obtains maximum directional derivative.
   (c) (4 pts) Calculate the directional derivative $\frac{\partial f}{\partial \mathbf{u}}(P)$, where $\mathbf{u}$ is a unit vector making an angle $\frac{\pi}{6}$ with the gradient $\nabla f(P)$.
   (d) (3 pts) The level surface $f(x,y,z) = \frac{\sqrt{3}}{2}$ defines z implicitly as a function of x and y near P. Compute $\frac{\partial z}{\partial x}$ at P.

Solution:

(a)

$$\begin{aligned}\nabla f &= (y\cos(xy+z), x\cos(xy+z), \cos(xy+z)) \\ &= \cos(xy+z)(y,x,1)\text{(each component 2\%)}\end{aligned}$$

(b) $\nabla f(0,-2,\frac{\pi}{3}) = \frac{1}{2}(-2,0,1)$ (each component $\frac{2}{3}$%)

(c)

$$\frac{\partial f}{\partial \vec{u}}(P) = \nabla f(P)\cdot u(2\%) = |\nabla f(P)|\cdot|u|\cdot\cos\frac{\pi}{6}(1\%) = \frac{\sqrt{5}}{2}\frac{\sqrt{3}}{2} = \frac{\sqrt{15}}{4}(1\%).$$

(d)

$$\begin{aligned}\sin(xy+z(x,y)) = \frac{\sqrt{3}}{2} &\Rightarrow \cos(xy+z(x,y))(y+z_x) = 0(2\%) \\ &\Rightarrow \cos(\frac{\pi}{3})(-2+z_x) = 0 \Rightarrow z_x = 2(1\%)\end{aligned}$$

2. (12 pts) Assume that $f(x, y, z)$ and $g(x, y, z)$ have continuous partial derivatives and $(1, 2, -1)$ lies on the level surface $f(x, y, z) = 3$. Suppose the tangent plane of $f(x, y, z) = 3$ at $(1, 2, -1)$ is $2x - y + 3z + 3 = 0$ and $f_y(1, 2, -1) = 2$.

(a) (4 pts) Find $\nabla f(1, 2, -1)$.

(b) (4 pts) Estimate $f(1.1, 2.01, -0.98)$ by the linear approximation of f at $(1, 2, -1)$.

(c) (4 pts) Suppose that when restricted on the surface $f(x, y, z) = 3$, $g(x, y, z)$ obtains maximum value at $(1, 2, -1)$ and $g_x(1, 2, -1) = -2$. Find $\nabla g(1, 2, -1)$ and the maximum directional derivative of g at the point $(1, 2, -1)$.

Solution:

(a) The tangent plane of $f(x, y, z) = 3$ at $(1, 2, -1)$ is

$$f_x(1, 2, -1)(x - 1) + f_y(1, 2, -1)(y - 2) + f_z(1, 2, -1)(z + 1) = 0. \quad \text{(2 pts)}$$

Since the tangent plane is $2x - y + 3z + 3 = 0$, $\nabla f(1, 2, -1) \mathbin{/\!/} (2, -1, 3)$. (1 pt)
Because $f_y(1, 2, -1) = 2$, we know that $\nabla f(1, 2, -1) = -2(2, -1, 3) = (-4, 2, -6)$. (1 pt).

$$\text{Ans: } \nabla f(1, 2, -1) = (-4, 2, -6).$$

(b) The linear approximation of f at $(1, 2, -1)$ is

$$L(x, y, z) = f(1, 2, -1) + f_x(1, 2, -1)(x - 1) + f_y(1, 2, -1)(y - 2) + f_z(1, 2, -1)(z + 1)$$

(2 pts for the definition of $L(x, y, z)$.)

$f(1.1, 2.01, -0.98) \approx L(1.1, 2.01, -0.98)$
$= 3 - 4(0.1) + 2(0.01) - 6(0.02)$ (1 pt for plugging in correct partial derivatives and x, y, z)
$= 2.5$ (1 pt for computation)

(c) By the method of Lagrange multiplies, we know that $\nabla g(1, 2, -1) = \lambda \nabla f(1, 2, -1)$ (1 pt)
Hence $\nabla g(1, 2, -1) = \lambda(-4, 2, -6)$.
By $g_x(1, 2, -1) = -2$, we know that $\nabla g = \frac{1}{2}(-4, 2, -6) = (-2, 1, -3)$ (1 pt).
The maximum directional derivative of g at $(1, 2, -1)$ is $|\nabla g(1, 2, -1)| = \sqrt{14}$
(1 pt for knowing the maximum directional derivative is $|\nabla g|$. 1 pt for the final answer.)

3. (25 pts) $f(x,y) = x^2 + xy + y^2 + 3x$.

(a) (7 pts) Find critical point(s) of $f(x,y)$ and determine whether it is a saddle point or $f(x,y)$ obtains local maximum or local minimum at it.

(b) (15 pts) Find the maximum and minimum value of $f(x,y)$ on the curve $x^2 + y^2 = 9$ by the method of Lagrange multipliers.

(c) (3 pts) Find the maximum value of $f(x,y)$ on the region $x^2 + y^2 \le 9$.

Solution:

(a) To find critical points of $f(x,y)$, we solve $\begin{cases} f_x = 2x + y + 3 = 0 \\ f_y = x + 2y = 0 \end{cases}$

(1 pt for setting $f_x = f_y = 0$. 1 pt for correct f_x and f_y.)

The solution is $(x,y) = (-2,1)$. (1 pt)

At $(-2,1)$, $f_{xx} = 2$, $f_{xy} = 1$, $f_{yy} = 2$ (1 pt)

$D(-2,1) = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{vmatrix}(-2,1) = \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} = 3$ (1 pt)

$\because D(-2,1) = 3 > 0$ and $f_{xx}(-2,1) = 2 > 0$ (1 pt)

We know that $f(-2,1)$ is a local minimum. (1 pt)

(b) By the method of Lagrange multipliers, to find extreme values of $f(x,y)$ on the constraint $g(x,y) = x^2 + y^2 = 9$, we solve $\begin{cases} \nabla f = \lambda \nabla g \\ g(x,y) = 9 \end{cases}$ (2 pts)

which is $\begin{cases} 2x + y + 3 = 2\lambda x & (1) \\ x + 2y = 2\lambda y & (2) \\ x^2 + y^2 = 9 & (3) \end{cases}$ (2 pts)

Method 1:

(2) $\Rightarrow x = 2(\lambda - 1)y$

plug in (1) $\Rightarrow (1 - 4(\lambda-1)^2)y = -3 \Rightarrow y = \frac{3}{4(\lambda-1)^2-1}$, $x = \frac{6(\lambda-1)}{4(\lambda-1)^2-1}$. (3 pts)

plug in (3) $\Rightarrow \frac{9+36(\lambda-1)^2}{(4(\lambda-1)^2-1)^2} = 9$

Let $(\lambda-1)^2 = u$, we have $1 + 4u = (4u-1)^2 \Rightarrow u = 0$ or $\frac{3}{4}$ i.e. $\lambda = 1$ or $1 \pm \frac{\sqrt{3}}{2}$ (3 pts)

when $\lambda = 1$, $y = -3$, $x = 0$

when $\lambda = 1 \pm \frac{\sqrt{3}}{2}$, $y = \frac{3}{2}$, $x = \pm\frac{3\sqrt{3}}{2}$

Critical points are $(x,y) = (0,-3)$, $(\frac{3\sqrt{3}}{2}, \frac{3}{2})$, and $(-\frac{3\sqrt{3}}{2}, \frac{3}{2})$. (3 pts)

$f(0,-3) = 9$, $f(\frac{3\sqrt{3}}{2}, \frac{3}{2}) = 9 + \frac{27\sqrt{3}}{4}$, $f(-\frac{3\sqrt{3}}{2}, \frac{3}{2}) = 9 - \frac{27\sqrt{3}}{4}$.

$$f(\frac{-3\sqrt{3}}{2}, \frac{3}{2}) < f(0,-3) < f(\frac{3\sqrt{3}}{2}, \frac{3}{2}).$$

Hence the maximum value is $f(\frac{3\sqrt{3}}{2}, \frac{3}{2}) = 9 + \frac{27\sqrt{3}}{4}$.

The minimum value is $f(-\frac{3\sqrt{3}}{2}, \frac{3}{2}) = 9 - \frac{27\sqrt{3}}{4}$. (2 pts)

Method 2:

Note that if $\lambda = 0$, then (1) and (2) $\Rightarrow (x,y) = (2,-1)$ but (3) is not satisfied. If $y = 0$, then (2) implies that x is also 0, but (3) is not satisfied for $(x,y) = (0,0)$.

Hence we conclude that $\lambda \neq 0$ and $y \neq 0$.

Therefore both sides of (2) are not zero.

Then we can divide $\frac{(1)}{(2)}$.

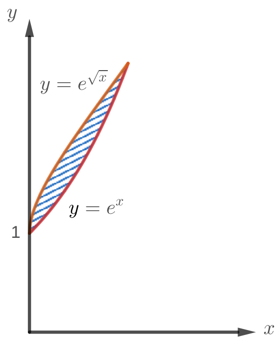
$$\frac{(1)}{(2)} \Rightarrow \frac{2x+y+3}{x+2y} = \frac{x}{y} \Rightarrow 2xy + y^2 + 3y = x^2 + 2xy$$
$$\Rightarrow x^2 = y^2 + 3y \text{ plug in } (3) \ \Rightarrow 2y^2 + 3y = 9$$
$$\Rightarrow y = \frac{3}{2} \text{ or } -3.$$

If $y = \frac{3}{2}$, $x = \pm\frac{3\sqrt{3}}{2}$. If $y = -3$, $x = 0$.
Hence critical points are $(x, y) = (0, -3)$, $(\frac{3\sqrt{3}}{2}, \frac{3}{2})$ and $(-\frac{3\sqrt{3}}{2}, \frac{3}{2})$.

(c) Inside the disc $x^2 + y^2 < 9$, $f(x, y,)$ has a critical point $(-2, 1)$. $f(-2, 1) = -3$.
On the boundary $x^2 + y^2 = 9$, $f(x, y)$ obtains maximum $f(\frac{3\sqrt{3}}{2}, \frac{3}{2}) = 9 + \frac{27\sqrt{3}}{4}$.
because $f(\frac{3\sqrt{3}}{2}, \frac{3}{2}) > f(-2, 1)$ (or $\because f(-2, 1)$ is a local minimum) (2 pts)
$\therefore$ The maximum value of f on the region $x^2 + y^2 \leq 9$ is $f(\frac{3\sqrt{3}}{2}, \frac{3}{2}) = 9 + \frac{27\sqrt{3}}{4}$. (1 pt)

4. (18 pts) (a) (8 pts) Reverse the order of integration and evaluate it. $\int_0^4 \int_{\frac{\sqrt{y}}{2}}^1 \sqrt{x^3+3}\,dx\,dy$.

(b) (10 pts) Compute $\iint\limits_{\Omega} (\ln y)^{-1} dA$, where Ω is bounded by $y = e^x$ and $y = e^{\sqrt{x}}$.



Solution:

(a)

$$\int_0^4 \int_{\sqrt{y}/2}^1 \sqrt{x^3+3}\,dx\,dy = \int_0^1 \int_0^{4x^2} \sqrt{x^3+3}\,dy\,dx \text{(4\%)}$$
$$= \int_0^1 4x^2\sqrt{x^3+3}\,dx \text{(2\%)}$$
$$= \frac{4}{3}\cdot\frac{2}{3}(x^3+3)^{3/2}\Big|_0^1 \text{(1\%)}$$
$$= \frac{8}{9}\left(8-3\sqrt{3}\right) \text{(1\%)}$$
$$= \frac{64}{9} - \frac{8}{3}\sqrt{3}$$

(b)

$$y = e^x \Rightarrow x = \ln y,$$
$$y = e^{\sqrt{x}} \Rightarrow \sqrt{x} = \ln y \Rightarrow x = (\ln y)^2.$$

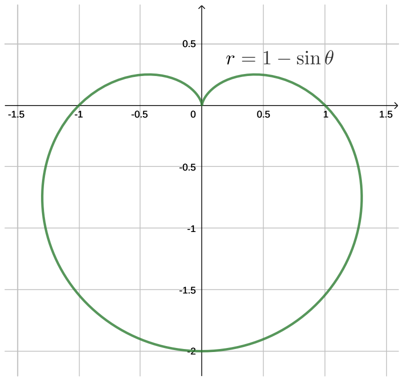
Then

$$\int\int_{\Omega} (\ln y)^{-1}\,dA = \int_1^e \int_{(\ln y)^2}^{\ln y} (\ln y)^{-1}\,dx\,dy \text{(4\%)}$$
$$= \int_1^e (\ln y)^{-1}\left(\ln y - \ln^2 y\right)\,dy \text{(3\%)}$$
$$= \int_1^e (1-\ln y)\,dy \text{(2\%)}$$
$$= (e-1) - (y\ln y - y)\Big|_1^e \text{(1\%)}$$
$$= e - 2.$$

5. (18 pts) (a) (8 pts) Evaluate $\iint\limits_D e^{-x^2-y^2} dA$, where D is the upper disc, $x^2+y^2 \le 25$ and $y \ge 0$.

(b) (10 pts) Calculate the area of the region inside the cardioid

$$r = 1 - \sin\theta.$$



Solution:

(a) Note that

$$D = \{[r,\theta] :\ 0 \le r \le 5,\ 0 \le \theta \le \pi\} \quad (2\%),$$

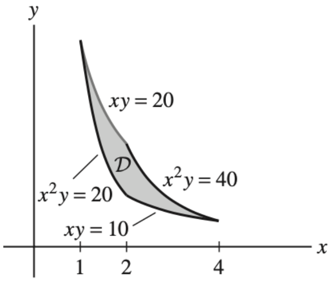
we have

$$\begin{aligned}\iint_D e^{-(x^2+y^2)}\, dA &= \int_0^\pi \int_0^5 e^{-r^2} \cdot r\, dr\, d\theta \quad (4\%)\\ &= \int_0^\pi \left(-\frac{e^{-r^2}}{2}\right)\Big|_0^5 d\theta \quad (1\%)\\ &= \pi(1-e^{-25})/2 \quad (1\%).\end{aligned}$$

(b)

$$\begin{aligned}\text{Area} = \iint_\Omega 1\, dA &= \int_0^{2\pi} \int_0^{1-\sin\theta} r\, dr\, d\theta \quad (4\%)\\ &= \int_0^{2\pi} \frac{(1-\sin\theta)^2}{2}\, d\theta \quad (1\%)\\ &= \int_0^{2\pi} \frac{1-2\sin\theta+\sin^2\theta}{2}\, d\theta \quad (1\%)\\ &= \int_0^{2\pi} \left(\frac{1}{2} - \sin\theta + \frac{1-\cos 2\theta}{4}\right) d\theta \quad (2\%)\\ &= \frac{3}{4}\cdot 2\pi = \frac{3\pi}{2} \quad (2\%).\end{aligned}$$

6. (12 pts) Evaluate $\iint_D e^{xy}\,dxdy$, where D is bounded by curves $xy = 10$, $xy = 20$, $x^2y = 20$ and $x^2y = 40$.



Solution:

Method 1: Let $u = xy$ and $v = x^2y$. Then $x = v/u$ and $y = u^2/v$ (2%). Further, (x, y) maps

$$\Omega = \{(u, v) : 10 \le u \le 20,\ 20 \le v \le 40\}$$

to D (2%). Since

$$J(u, v) = \begin{vmatrix} -v/u^2 & 2u/v \\ 1/u & -u^2/v^2 \end{vmatrix} = -1/v \quad (2\%),$$

we have

$$\begin{aligned} \iint_D e^{xy}\,dx\,dy &= \int_{10}^{20}\int_{20}^{40} e^u(|-v^{-1}|)\,du\,dv \quad (4\%) \\ &= \int_{10}^{20} e^u\,du \cdot \int_{20}^{40} v^{-1}\,dv \\ &= (e^{20} - e^{10})\ln 2 \quad (2\%). \end{aligned}$$

Method 2: Let $u = xy$ and $v = x^2y$ (2%). Then (u, v) maps D to

$$\Omega = \{(u, v) : 10 \le u \le 20,\ 20 \le v \le 40\} \quad (2\%).$$

Since

$$J(x, y) = \begin{vmatrix} y & 2xy \\ x & x^2 \end{vmatrix} = -x^2y = -v \quad (1\%),$$

we have

$$J(u, v) = J(x, y)^{-1} = -v^{-1} \quad (1\%).$$

We have,

$$\begin{aligned} \iint_D e^{xy}\,dx\,dy &= \int_{10}^{20}\int_{20}^{40} e^u(|-v^{-1}|)\,du\,dv \quad (4\%) \\ &= \int_{10}^{20} e^u\,du \cdot \int_{20}^{40} v^{-1}\,dv \\ &= (e^{20} - e^{10})\ln 2 \quad (2\%). \end{aligned}$$