1092 Calculus B 01-03 Final Exam Solution

June 24, 2021

1. Consider the differential equation $y\dfrac{dy}{dt} = \sqrt{1-y^2}$.
   (a) (6 pts) If the constant function $y(t) = a$ is a solution of the equation, find the value of a.
   (b) (10 pts) Given that $y(0) = \dfrac{1}{2}$, solve $y(t)$ for t near 0.

   考慮微分方程式 $y\dfrac{dy}{dt} = \sqrt{1-y^2}$。
   (a) (6 pts) 如果常數函數 $y(t) = a$ 是方程式的解，求a 的值。
   (b) (10 pts) 給定 $y(0) = \dfrac{1}{2}$，求 $y(t)$， 當t在0附近。

Solution:

(a) Let $\sqrt{1-y^2} = 0$. Then $y = \pm 1$. Hence

$$a = \pm 1 \quad (6\%).$$

(b) Now suppose $y \neq \pm 1$ (1%). Rewrite the equation as

$$\frac{y\,dy}{\sqrt{1-y^2}} = dt \quad (1\%)$$

and integrate on both sides with respect to t, we get

$$-\sqrt{1-y^2} = x + c. \quad (3\%).$$

Since $y(0) = 1/2$, we get that

$$c = -\sqrt{1-(1/2)^2} = -\frac{\sqrt{3}}{2} \quad (2\%).$$

Hence

$$\begin{aligned} y &= \pm\sqrt{1-(x-\sqrt{3}/2)^2} \quad (1\%) \\ &= \sqrt{1-(x-\sqrt{3}/2)^2} \quad \left(\text{since } y(0) = \frac{1}{2}, \text{ we should choose } +\right) \quad (2\%). \end{aligned}$$

2. (16 pts) Solve

$$\begin{cases} y'(t) = t(t^2 - y(t)), \\ y(0) = 3. \end{cases}$$

(16 pts) 解

$$\begin{cases} y'(t) = t(t^2 - y(t)), \\ y(0) = 3. \end{cases}$$

Solution:

$y' + t\,y = t^3$	rearrange the eqn
$I(t) = e^{\int t\,dt} = e^{\frac{1}{2}t^2}$	find IF (5%)
$e^{\frac{1}{2}t^2}\,y' + e^{\frac{1}{2}t^2}\,t\,y = e^{\frac{1}{2}t^2}\,t^3$	multiply the eqn by $I(t) = e^{\frac{1}{2}t^2}$ (2%)
$\dfrac{d\left(e^{\frac{1}{2}t^2}\,y\right)}{dt} = e^{\frac{1}{2}t^2}\,t^3$	rewrite the left-hand side
$\displaystyle\int \frac{d\left(e^{\frac{1}{2}t^2}\,y\right)}{dt}\,dt = \int e^{\frac{1}{2}t^2}\,t^3\,dt$	integrate both sides (2%)
$e^{\frac{1}{2}t^2}\,y = e^{\frac{1}{2}t^2}(t^2 - 2) + c$	evaluate the integrals (3%)
$3 = -2 + c$	use $y(0) = 3$
$5 = c$	find c (2%)
$y(t) = t^2 - 2 + 5\,e^{\frac{-t^2}{2}}$	the solution (2%)

3. (8 pts) Suppose that X and Y are independent and $\mathrm{E}(X) = 1$, $\mathrm{E}(Y) = 2$, $\mathrm{Var}(X) = 5$, $\mathrm{Var}(Y) = 7$. Compute $\mathrm{E}((2X - Y)^2)$.
(8 pts) 已知 X，Y 爲兩個獨立的隨機變數，而且 $\mathrm{E}(X) = 1$，$\mathrm{E}(Y) = 2$， $\mathrm{Var}(X) = 5$， $\mathrm{Var}(Y) = 7$。求 $\mathrm{E}((2X - Y)^2)$。

Solution:

$\mathrm{E}((2X`Y)^2)$

$=\mathrm{E}(4X^2 - 4XY + Y^2)$ expand (2%)

$=4\mathrm{E}(X^2) - 4\mathrm{E}(XY) + \mathrm{E}(Y^2)$ $\mathrm{E}(aX + bY) = aE(X) + bE(Y)$ (1%)

$=4(\mathrm{Var}(X) + (\mathrm{E}(X))^2) - 4\mathrm{E}(X) \cdot \mathrm{E}(Y)$ $\mathrm{Var}(X) = \mathrm{E}(X^2) - (\mathrm{E}(X))^2$ (2%)

$+ (\mathrm{Var}(Y) + (\mathrm{E}(Y))^2)$ X, Y idep. $\Rightarrow \mathrm{E}(XY) = \mathrm{E}(X) \cdot \mathrm{E}(Y)$ (1%)

$=4(5 + 1^2) - 4 \cdot 1 \cdot 2 + (7 + 2^2)$ input the values (2%)

$=27$

4. Suppose that X is a random variable with probability density function $f_X(t) = ce^{\frac{-t^2+8t}{20}}$ for some constant $c > 0$.

(a) (8 pts) Find the constant c such that $\int_{-\infty}^{\infty} f_X(t)dt = 1$. (Hint: $\int_{-\infty}^{\infty} e^{-t^2} dt = \sqrt{\pi}$)

(b) (8 pts) Find $\mathrm{E}(X)$ and $\mathrm{Var}(X)$.

(c) (6 pts) Suppose that $X_1, \cdots, X_n$ are independent random variables and $X_i \sim X$ for $1 \le i \le n$. Use Chebyshev's inequality to estimate the size of n such that we can derive

$$P(\left|\frac{X_1 + \cdots + X_n}{n} - \mathrm{E}(X)\right| < 0.05\ ) > 0.9.$$

已知隨機變數 X 的機率密度函數爲 $f_X(t) = ce^{\frac{-t^2+8t}{20}}$，其中 $c > 0$ 是一個常數。

(a) (8 pts) 求常數 c 使得 $\int_{-\infty}^{\infty} f_X(t)dt = 1$. (提示: $\int_{-\infty}^{\infty} e^{-t^2} dt = \sqrt{\pi}$)

(b) (8 pts) 求 $\mathrm{E}(X)$ 和 $\mathrm{Var}(X)$.

(c) (6 pts) 假設 $X_1, \cdots, X_n$ 爲互相獨立的隨機變數而且 $X_i \sim X$， $1 \le i \le n$。 利用 Chebyshev 不等式，估計 n 需要多大，我們才能保證

$$P(\left|\frac{X_1 + \cdots + X_n}{n} - \mathrm{E}(X)\right| < 0.05\ ) > 0.9.$$

Solution:

(a) We have

$$\begin{aligned} 1 &= c\int_{-\infty}^{\infty} e^{-(t^2-8t)/20}\, dt \\ &= c\int_{-\infty}^{\infty} e^{-\frac{(t-4)^2-16}{20}}\, dt \quad (3\%) \\ &= ce^{\frac{4}{5}} \int_{-\infty}^{\infty} e^{-\frac{(t-4)^2}{20}}\, dt \quad (1\%) \\ &= ce^{\frac{4}{5}} \cdot \sqrt{\pi} \cdot \sqrt{20} \quad (3\%). \end{aligned}$$

Hence

$$c = e^{-4/5} \cdot \frac{1}{2\sqrt{5\pi}} \quad (1\%).$$

(b) Follows form (a), we have that

$$X \sim \mathrm{N}(4, 10) \quad (2\%).$$

Hence

$$\mathrm{E}(X) = 4 \quad (3\%), \quad \mathrm{Var}(\mathrm{X}) = 10 \quad (3\%).$$

(c) Let

$$Z := \frac{X_1 + \cdots + X_n}{n}.$$

Since $X_i \sim X$ for all i and $X_1, \cdots, X_n$ are independent, we have

$$\mathrm{E}(Z) = \mathrm{E}(X) = 4 \quad (1\%) \quad \mathrm{Var}(Z) = \frac{\mathrm{Var}(X)}{n} = \frac{10}{n} \quad (1\%).$$

By the Chebyshev inequality, we want to find n so that

$$\begin{aligned} P\left(\left|Z - \mathrm{E}(X)\right| < 0.05\right) &\geq 1 - \frac{\mathrm{Var(Z)}}{(0.05)^2} \quad (2\%) \\ &= 1 - \frac{10/n}{(0.05)^2} \quad (1\%) \\ &> 0.9, \end{aligned}$$

this is equivalent to

$$0.1 > \frac{10/n}{1/400} \iff n > 40000 \quad (1\%).$$

5. Let $f(t) = \begin{cases} c \cdot te^{-t}, & t > 0 \\ 0, & t \leq 0. \end{cases}$

(a) (8 pts) Find constant c such that $f(t)$ is a probability density function.

(b) (4 pts) Suppose that X and Y are independent with probability density functions $f_X = f_Y = f$. Let $Z = X + Y$. Find the plane region D in the xy-plane such that the distribution function of Z, $F_Z(t) = P(X + Y \leq t)$, is $\iint_D f_X(x) f_Y(y) dxdy$.

(c) (10 pts) Find the probability density function of Z in part (b).

令 $f(t) = \begin{cases} c \cdot te^{-t}, & t > 0 \\ 0, & t \leq 0. \end{cases}$

(a) (8 pts) 求常數 c 使得 $f(t)$ 是一個機率密度函數。

(b) (4 pts) 已知隨機變數 X 和 Y 是獨立的而且它們的機率密度函數爲 $f_X = f_Y = f$。令 $Z = X+Y$。求 xy 平面的區域D，使得 Z的分配函數， $F_Z(t) = P(X+Y \leq t)$， 是重積分 $\iint_D f_X(x) f_Y(y) dxdy$。

(c) (10 pts) 求(b)小題中 Z的機率密度函數。

Solution:

(a)

$$\int_{-\infty}^{\infty} f(t)dt = c\int_0^{\infty} te^{-t}dt = c\left(-te^{-t}\Big|_0^{\infty} + \int_0^{\infty} e^{-t}dt\right) = c(0 - e^{-t}\Big|_0^{\infty}) = c.$$

(3 pts for applying integration by parts. 3 pts for correct result. Students do not need to use the definition of improper integrals to compute this integration.)

Let $c = 1$. Then $\int_{-\infty}^{\infty} f(t)dt = 1$ and $f(t)$ is a probability density function.

(2 pts for $c = 1$.)

(b) $F_Z(t) = P(X + Y \leq t) = \iint_D f_X(x) f_Y(y) dxdy$, where $D = \{(x,y) | x + y \leq t, x \geq 0, y \geq 0\}$.

If $t \leq 0$, then D is an empty set or a point $\{(0,0)\}$.

If $t > 0$, then D is the triangle bounded by $x = 0$, $y = 0$, and $x + y = t$.

(4 pts for $D = \{(x,y) | x + y \leq t, x \geq 0, y \geq 0\}$ or $D = \{(x,y) | x + y \leq t\}$. It is o.k. if students don't discuss cases $t > 0$ and $t \leq 0$.)

(c) For $t \leq 0$, $F_Z(t) = 0$

For $t > 0$,

Solution 1:

$$F_Z(t) = \iint_D xe^{-x} ye^{-y} dxdy \xlongequal{\begin{cases} u = x + y \\ v = y \end{cases}} \iint_\Omega (u - v)e^{v-u} \cdot v \cdot e^{-v} \left|\frac{\partial(x,y)}{\partial(u,v)}\right| dudv$$

$$= \int_0^t \int_0^u (u - v)v \cdot e^{-u} dvdu,$$

where Ω is the region in the uv-plane bounded by $u = t$, $u = v$, and $v = 0$, and $\left|\frac{\partial(x,y)}{\partial(u,v)}\right| = 1$.

(1 pt for substitution $u = x + y$, $v = y$, or $u = x + y$, $v = x$. 1 pt for $\left|\frac{\partial(x,y)}{\partial(u,v)}\right| = 1$. 1 pt for Ω. 2 pts for the iterated integral.)

Hence $f_Z(t) = F_Z'(t) = \int_0^t e^{-t} \cdot (t-v)vdv = \frac{1}{6}t^3 e^{-t}$.

(2 pts for $f_Z(t) = F_Z'(t) = \int_0^t e^{-t} \cdot (t-v)vdv$. 3 pts for $\int_0^t e^{-t} \cdot (t-v)vdv = \frac{1}{6}t^3 e^{-t}$.)

Solution 2:

$f_Z(t) = \int_{-\infty}^{\infty} f_X(t-s) f_Y(s)\, ds$ or $\int_{-\infty}^{\infty} f_X(s) f_Y(t-s)\, ds$. (3 pts)

Hence $f_Z(t) = \int_{-\infty}^{\infty} f_X(t-s) f_Y(s)\, ds = \int_0^t (t-s)e^{s-t} s e^{-s}\, ds$

(2 pts for the upper and lower limits of the integration $0, t$.)

Thus $f_Z(t) = \int_0^t s(t-s)e^{-t}\, ds \ = \ \frac{1}{6}t^3 e^{-t}$.

(2 pt for $f_Z(t) = \int_0^t s(t-s)e^{-t}\, ds$. 3 pts for $\int_0^t s(t-s)e^{-t}\, ds \ = \ \frac{1}{6}t^3 e^{-t}$.)

6. The number of flaws in a carpet appears to be Poisson distributed at a rate of one every 6m^2.

(a) (8 pts) Find the probability that a 3m by 4m carpet contains more than 2 flaws.

(b) (8 pts) There are two carpets with sizes 2m by 4m and 2m by 5m. Find the probability that these two carpets together contain less than 2 flaws.

假設地毯上的瑕疵個數呈 Poisson 分配，而且平均每 6平方公尺有一個瑕疵。

(a) (8 pts) 求一條 3公尺寬 4公尺長的地毯有超過 2個瑕疵的機率。

(b) (8 pts) 有兩條地毯，各是 2公尺寬 4公尺長和 2公尺寬 5公尺長。求這兩條地毯上總共的瑕疵數少於 2的機率。

Solution:

(a) The average number of flaws in a 3m×4m carpet is $1 \times (\frac{3 \times 4}{6}) = 2$. (2 pts)
Let F be the number of flaws in the carpet. Then F is Poisson distributed with $E(F) = 2$.
Hence $P(F = 0) = e^{-2}$, $P(F = 1) = \frac{2^1}{1!}e^{-2} = 2e^{-2}$, $P(F = 2) = \frac{2^2}{2!}e^{-2=2e^{-2}}$.
(1 pt for $P(F = 0) = e^{-2}$. 1 pt for $P(F = 1) = 2e^{-2}$. 1 pt for $P(F = 2) = 2e^{-2}$.)
Thus $P(F > 2) = 1 - P(F \le 2) = 1 - P(F = 0) - P(F = 1) - P(F = 2) = 1 - 5 \cdot e^{-2}$.
(2 pts for $P(F > 2) = 1 - P(F \le 2) = 1 - P(F = 0) - P(F = 1) - P(F = 2)$. 1 pt for the final answer.)

(b) **Solution 1:**

Together the average number of flaws in the two carpets is $1 \times \frac{(2 \times 4 + 2 \times 5)}{6} = 3$. (4 pts)
Let F be the number of flaws in the two carpets. Then $P(F < 2) = P(F = 0) + P(F = 1) = e^{-3} + 3e^{-3} = 4e^{-3}$.
(1 pt for $P(F < 2) = P(F = 0) + P(F = 1)$. 1 pt for $P(F = 0) = e^{-3}$. 1 pt for $P(F = 1) = 3e^{-3}$. 1 pt for the final answer.)

Solution 2: Let F_1 be the number of flaws in the 2×4 carpet. $P(F_1 = 0) = e^{-\frac{8}{6}}$ and $P(F_1 = 1) = \frac{4}{3}e^{-\frac{4}{3}}$. (2 pts)

Let F_2 be the number of flaws in the 2×5 carpet. $P(F_2 = 0) = e^{\frac{-10}{6}}$, $P(F_2 = 1) = \frac{5}{3}e^{-\frac{5}{3}}$. (2 pts)

$$P(F_1 + F_2 \le 1) = P(F_1 = 0, F_2 = 0) + P(F_1 = 1, F_2 = 0) + P(F_1 = 0, F_2 = 1) \text{ (2 pts)}$$
$$= e^{-\frac{4}{3}}e^{-\frac{5}{3}} + \frac{4}{3}e^{-\frac{4}{3}}e^{-\frac{5}{3}} + e^{-\frac{4}{3}}\frac{5}{3}e^{-\frac{5}{3}} = 4e^{-3}. \text{ (2 pts)}$$