1. (a) (5%) Show that $\int_0^{\frac{\pi}{2}} e^x \csc x\,dx$ diverges to infinity by the comparison test.

   (b) (5%) Evaluate the limit $\lim_{t\to 0+} t \int_t^{\frac{\pi}{2}} e^x \csc x\,dx$.

Solution:

(a) Since

$$e^x \csc x = \frac{e^x}{\sin x} > \frac{1}{x} > 0 \quad \forall\ 0 < x \le \frac{\pi}{2}, \qquad \textbf{(3 points)}$$

the comparison test for improper integrals implies

$$\int_0^{\frac{\pi}{2}} e^x \csc x\,dx \ge \int_0^{\frac{\pi}{2}} \frac{1}{x}\,dx = \infty. \qquad \textbf{(2 points)}$$

(b) Note that

$$\lim_{t\to 0+} t \int_t^{\frac{\pi}{2}} e^x \csc x\,dx = \lim_{t\to 0+} \frac{\int_t^{\frac{\pi}{2}} e^x \csc x\,dx}{t^{-1}} \qquad \textbf{(1 point)}$$

is an indeterminate form of type $\frac{\infty}{\infty}$. Applying L'Hospital's rule gives

$$\lim_{t\to 0+} \frac{\int_t^{\frac{\pi}{2}} e^x \csc x\,dx}{t^{-1}} = \lim_{t\to 0+} \frac{-e^t \csc t}{-t^{-2}} \qquad \textbf{(2 points)}$$

$$= \lim_{t\to 0+} \frac{t^2 e^t}{\sin t} = \lim_{t\to 0+} \left(te^t \frac{t}{\sin t}\right) = 0. \qquad \textbf{(2 points)}$$

2. Evaluate the following integrals.

(a) (7%) $\int x \cdot \tan^{-1}\sqrt{x}\,dx$.

(b) (8%) $\int \frac{x^3-3x^2+4x-1}{(x^2-2x+2)^2}\,dx$.

(c) (7%) $\int_0^{\pi/2} \frac{1}{3+(\sin\theta)^2}\,d\theta$. (**Hint.** Consider the substitution $t=\tan\theta$.)

Solution:

(a) **Solution 1:**

$$\begin{aligned}
\int x\cdot\tan^{-1}\sqrt{x}\,dx &= \frac{x^2}{2}\tan^{-1}\sqrt{x} - \int \frac{x^2}{2}\frac{1}{1+x^2}\frac{1}{2\sqrt{x}}\,dx && \text{(2 pts)}\\
&\overset{u=\sqrt{x}}{\underset{2udu=dx}{=\!=\!=}} \frac{1}{2}x^2\tan^{-1}\sqrt{x} - \frac{1}{2}\int \frac{u^4}{1+u^2}du && \text{(2 pts)}\\
&= \frac{1}{2}x^2\tan^{-1}\sqrt{x} - \frac{1}{2}\int u^2-1+\frac{1}{1+u^2}du && \text{(1 pt)}\\
&= \frac{1}{2}x^2\tan^{-1}\sqrt{x} - \frac{1}{6}u^3+\frac{1}{2}u-\frac{1}{2}\tan^{-1}u+c\\
&= \frac{1}{2}x^2\tan^{-1}\sqrt{x} - \frac{1}{6}x\sqrt{x}+\frac{1}{2}\sqrt{x}-\frac{1}{2}\tan^{-1}\sqrt{x}+c && \text{(2 pts)}
\end{aligned}$$

Solution 2:

$$\begin{aligned}
\int x\cdot\tan^{-1}\sqrt{x}\,dx &\overset{u=\sqrt{x}}{\underset{2udu=dx}{=\!=\!=}} 2\int u^3\tan^{-1}u\,du && \text{(2 pts)}\\
&= \frac{1}{2}u^4\tan^{-1}u - \frac{1}{2}\int \frac{u^4}{1+u^2}du && \text{(2 pts)}\\
&= \frac{1}{2}x^2\tan^{-1}\sqrt{x} - \frac{1}{2}\int u^2-1+\frac{1}{1+u^2}du && \text{(1 pt)}\\
&= \frac{1}{2}x^2\tan^{-1}\sqrt{x} - \frac{1}{6}x\sqrt{x}+\frac{1}{2}\sqrt{x}-\frac{1}{2}\tan^{-1}\sqrt{x}+c && \text{(2 pts)}
\end{aligned}$$

(b) $\frac{x^3-3x^2+4x-1}{(x^2-2x+2)^2} = \frac{(x-1)(x^2-2x+2)+1}{(x^2-2x+2)^2} = \frac{x-1}{x^2-2x+2}+\frac{1}{(x^2-2x+2)^2}$

(1 pt for correct form $\frac{ax+b}{x^2-2x+2}+\frac{cx+d}{(x^2-2x+2)^2}$. 2 pts for correct constants.)

$$\begin{aligned}
&\int \frac{x^3-3x^2+4x-1}{(x^2-2x+2)^2}dx = \int \frac{x-1}{x^2-2x+2}+\frac{1}{(x^2-2x+2)^2}dx\\
&\overset{u=x-1}{\underset{du=dx}{=\!=\!=}} \int \frac{u}{u^2+1}+\frac{1}{(u^2+1)^2}du && \text{(1 pt for completing squares and making substitution)}\\
&= \frac{1}{2}\ln(x^2-2x+2)+\int\frac{1}{(u^2+1)^2}du && \text{(1 pt for } \int\frac{u}{u^2+1}du=\frac{1}{2}\ln(x^2-2x+2)\text{)}\\
&\overset{u=\tan\theta,-\frac{\pi}{2}<\theta<\frac{\pi}{2}}{\underset{du=\sec^2\theta d\theta}{=\!=\!=}} \frac{1}{2}\ln(x^2-2x+2)+\int\frac{\sec^2\theta}{\sec^4\theta}d\theta && \text{(1 pt for trigonometric substitution)}\\
&= \frac{1}{2}\ln(x^2-2x+2)+\int\frac{1+\cos2\theta}{2}d\theta\\
&= \frac{1}{2}\ln(x^2-2x+2)+\frac{\theta}{2}+\frac{\sin2\theta}{4}+c\\
&= \frac{1}{2}\ln(x^2-2x+2)+\frac{1}{2}\tan^{-1}(x-1)+\frac{1}{2}\frac{x-1}{x^2-2x+2}+c
\end{aligned}$$

(2 pts for $\int\cos^2\theta d\theta = \frac{1}{2}\tan^{-1}(x-1)+\frac{1}{2}\frac{x-1}{x^2-2x+2}+c$)

(c) **Solution 1:**

Let $t=\tan\theta$, $dt=\sec^2\theta d\theta=(1+t^2)d\theta$ i.e. $d\theta=\frac{dt}{1+t^2}$ (1 pt)

$\sin^2\theta = 1 - \cos^2\theta = 1 - \dfrac{1}{1+\tan^2\theta} = 1 - \dfrac{1}{1+t^2}$ (1 pt)

$$\int_0^{\frac{\pi}{2}} \frac{1}{3+(\sin\theta)^2}d\theta = \int_0^{\infty} \frac{1}{3+1-\frac{1}{1+t^2}}\frac{1}{1+t^2}dt$$

(1 pt for the upper and lower bound of the integrals.)

$$= \int_0^{\infty} \frac{1}{4(1+t^2)-1}dt$$

$$= \int_0^{\infty} \frac{1}{4t^2+3}dt \qquad \text{(1 pt for simplifying the integrand.)}$$

$$= \frac{1}{3}\int_0^{\infty} \frac{1}{1+(\frac{2t}{\sqrt{3}})^2}dt$$

$$\overset{u=\frac{2}{\sqrt{3}}t}{\underset{du=\frac{2}{\sqrt{3}}dt}{=\!=\!=}} \frac{1}{2\sqrt{3}}\int_0^{\infty}\frac{1}{1+u^2}du \qquad \text{(2 pts)}$$

$$= \frac{1}{2\sqrt{3}}\lim_{t\to\infty}\left(\int_0^{t}\frac{1}{1+u^2}du\right) = \frac{\pi}{4\sqrt{3}} \qquad \text{(1 pt)}$$

Solution 2:

$$\int_0^{\frac{\pi}{2}} \frac{1}{3+(\sin\theta)^2}d\theta = \int_0^{\frac{\pi}{2}} \frac{\sec^2\theta}{3\sec^2\theta+\tan^2\theta}d\theta$$

$$= \int_0^{\frac{\pi}{2}} \frac{\sec^2\theta}{3+4\tan^2\theta}d\theta \qquad \text{(2 pts)}$$

$$\overset{t=\tan\theta}{\underset{dt=\sec^2\theta}{=\!=\!=}} \int_0^{\infty}\frac{dt}{3+4t^2} \qquad \text{(2 pts)}$$

$$= \frac{1}{3}\int_0^{\infty}\frac{1}{1+(\frac{2t}{\sqrt{3}})^2}dt \overset{u=\frac{2t}{\sqrt{3}}}{=\!=\!=} \frac{1}{2\sqrt{3}}\int_0^{\infty}\frac{1}{1+u^2}du \qquad \text{(2 pts)}$$

$$= \frac{\pi}{4\sqrt{3}} \qquad \text{(1 pt)}$$

3. Let $F(x) = \int_x^1 \frac{dt}{t\sqrt{t+1}}$.

(a) (2%) Compute $F'(x)$.

(b) (8%) Determine whether the improper integral $\int_0^1 \frac{1}{\sqrt{x}} F(x)\,dx$ converges or not. Evaluate the integral if it is convergent. (**Hint.** Use integration by parts.)

Solution:

(a) Given $x \in (0,1)$, since the integrand

$$\frac{1}{t\sqrt{t+1}}$$

is continuous on $[x,1]$, the fundamental theorem of calculus implies

$$F'(x) = \frac{-1}{x\sqrt{x+1}}. \qquad \textbf{(2 points)}$$

(b) Given $\epsilon \in (0,1)$, integration by parts gives

$$\int_\epsilon^1 \frac{1}{\sqrt{x}} F(x)\,dx = 2\int_\epsilon^1 F(x)\,d\sqrt{x} = 2F(x)\sqrt{x}\Big|_{x=\epsilon}^1 - 2\int_\epsilon^1 \sqrt{x}F'(x)\,dx \qquad \textbf{(2 points)}$$

$$= 2\sqrt{x}\int_x^1 \frac{dt}{t\sqrt{t+1}}\Bigg|_{x=\epsilon}^1 + 2\int_\epsilon^1 \frac{\sqrt{x}}{x\sqrt{x+1}}dx$$

$$= -2\sqrt{\epsilon}\int_\epsilon^1 \frac{dt}{t\sqrt{t+1}} + 2\int_\epsilon^1 \frac{\sqrt{x}}{x\sqrt{x+1}}dx. \qquad \textbf{(2 points)}$$

To compute the second term, let's use the substitution $u = \sqrt{x}$ to get

$$2\int_\epsilon^1 \frac{\sqrt{x}}{x\sqrt{x+1}}dx = 2\int_{\sqrt{\epsilon}}^1 \frac{u}{u^3+1}2u\,du = \frac{4}{3}\ln\left(u^3+1\right)\Big|_{u=\sqrt{\epsilon}}^1 = \frac{4}{3}\ln\left(\frac{2}{1+\epsilon^{\frac{3}{2}}}\right). \qquad \textbf{(2 points)}$$

Therefore,

$$\lim_{\epsilon\to 0+}\int_\epsilon^1 \frac{1}{\sqrt{x}}F(x)\,dx = \lim_{\epsilon\to 0+}\left(-2\sqrt{\epsilon}\int_\epsilon^1 \frac{dt}{t\sqrt{t+1}} + \frac{4}{3}\ln\left(\frac{2}{1+\epsilon^{\frac{3}{2}}}\right)\right) = \frac{4}{3}\ln 2. \qquad \textbf{(2 points)}$$

That is, the improper integral converges and is equal to $\frac{4}{3}\ln 2$.

4. (10%) A bowl is generated by rotating a plane region R about the y-axis, where R is bounded by lines $x = 0$, $y = 1$, and the curve $y = (\sin(x^2))^2$, $0 \le x \le \sqrt{\frac{\pi}{2}}$. Find the volume of the bowl.

Solution:

(1) In the range $0 \le x \le \sqrt{\pi/2}, 0 \le y \le 1$, the function $y = (\sin x^2)^2$ has the inverse $x = \sqrt{\arcsin \sqrt{y}}$ `(2 points)`. The volume is given by

$$\begin{aligned}\pi \int_0^1 x^2\,dy &= \pi \int_0^1 \sin^{-1}\sqrt{y}\,dy \quad \texttt{(3 points)}\\ &= \pi \int_0^1 2t\sin^{-1} t\,dt \quad (y = t^2)\\ &= \pi(t^2 \sin^{-1} t\Big|_0^1 - \int_0^1 \frac{t^2}{\sqrt{1-t^2}}\,dt) \quad (u = \sin^{-1} t, v' = 2t)\\ &= \pi(\frac{\pi}{2} + t\sqrt{1-t^2}\Big|_0^1 - \int_0^1 \sqrt{1-t^2}\,dt)\\ &\qquad\qquad (u = t, v' = \frac{t}{\sqrt{1-t^2}})\\ &= \pi(\frac{\pi}{2} - \frac{\pi}{4}) = \frac{\pi^2}{4}. \quad \texttt{(5 points)}\end{aligned}$$

(2) The volume equals

$$2\pi \int_0^{\sqrt{\pi/2}} x(1 - (\sin x^2)^2)\,dx. \quad \texttt{(5 points)}$$

The integral equals

$$\begin{aligned}\int_0^{\sqrt{\pi/2}} x\cos^2 x^2\,dx &= \frac{1}{2}\int_0^{\sqrt{\pi/2}} x(1 + \cos 2x^2)\,dx\\ &= \frac{1}{2}\Big(\frac{1}{2}x^2 + \frac{1}{4}\sin 2x^2\Big)\Big|_0^{\sqrt{\pi/2}}\\ &= \frac{1}{8}\pi \quad \texttt{(5 points).}\end{aligned}$$

5. (03-05班) Let C be the curve $y = \ln x$, $1 \le x \le \sqrt{3}$.

(a) (10%) Find the arc length of C.

(b) (10%) Rotate C about the y-axis. Find the area of the resulting surface.

Solution:

(a)

$$L = \int_1^{\sqrt{3}} \sqrt{1 + (\frac{dy}{dx})^2} dx \qquad \text{(2 pts)}$$

$$= \int_1^{\sqrt{3}} \frac{\sqrt{1+x^2}}{x} dx \qquad \text{(1 pt)}$$

$$\overset{x=\tan\theta, -\frac{\pi}{2}<\theta<\frac{\pi}{2}}{\underset{dx=\sec^2\theta d\theta}{=\!=\!=\!=}} \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{\sec\theta}{\tan\theta} \cdot \sec^2\theta d\theta \qquad \text{(2 pts)}$$

$$= \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{\sec^2\theta}{\tan^2\theta} \sec\theta \tan\theta \, d\theta$$

$$\overset{u=\sec\theta}{\underset{du=\sec\theta\tan\theta\, d\theta}{=\!=\!=\!=}} \int_{\sqrt{2}}^{2} \frac{u^2}{u^2-1} du \qquad \text{(2 pts)}$$

$$= \int_{\sqrt{2}}^{2} 1 + \frac{1}{2}\left(\frac{1}{u-1} - \frac{1}{u+1}\right) du \qquad \text{(1 pt for partial fraction)}$$

$$= 2 - \sqrt{2} + \frac{1}{2}(2\ln(\sqrt{2}+1) - \ln 3) \qquad \text{(1 pt for integration)}$$

(1 pt for final answer. Do not need to simplify the answer.

The answer is equivalent to $2 - \sqrt{2} + \frac{1}{2}\left(\ln\frac{1}{3} - \ln\frac{\sqrt{2}-1}{\sqrt{2}+1}\right)$)

(b)

$$S = \int 2\pi x dS = 2\pi \int_1^{\sqrt{3}} x\sqrt{1 + (\frac{dy}{dx})^2} dx \qquad \text{(2 pts)}$$

$$= 2\pi \int_1^{\sqrt{3}} x \frac{\sqrt{1+x^2}}{x} dx \qquad \text{(1 pt)}$$

$$= 2\pi \int_1^{\sqrt{3}} \sqrt{1+x^2} dx \overset{x=\tan\theta, -\frac{\pi}{2}<\theta<\frac{\pi}{2}}{\underset{dx=\sec^2\theta d\theta}{=\!=\!=\!=}} 2\pi \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \sec^3\theta \, d\theta \qquad \text{(2 pts)}$$

$$= \pi[\sec\theta\tan\theta + \ln|\sec\theta + \tan\theta|]\Big|_{\theta=\frac{\pi}{4}}^{\theta=\frac{\pi}{3}} \qquad \text{(4 pts for integrating } \int \sec^3\theta \, d\theta)$$

$$= \pi[2\sqrt{3} + \ln(2+\sqrt{3}) - \sqrt{2} - \ln(\sqrt{2}+1)]^{\text{(1 pt for final answer.)}}$$

(01-02班) (12 pts) Solve the following differential equations.

(a) (4%) $y'' - 2y' + y = 0$.

(b) (8%) $y'' - 2y' + y = \dfrac{e^x}{1+x^2}$.

Solution:

(a) The characteristic polynomial

$$r^2 - 2r + 1 = (r-1)^2 \quad \texttt{(2 points)}$$

of the homogeneous equation has a double root equal to 1. Thus the general solution is

$$y_c = c_1 e^x + c_2 x e^x. \quad \texttt{(2 points)}$$

(b) Via the method of variation of parameters, a particular solution is of the form

$$y_p = u_1 e^x + u_2 x e^x$$

satisfying

$$\begin{aligned} u_1' e^x + u_2' x e^x &= 0, \\ u_1'(e^x)' + u_2'(xe^x)' &= \frac{e^x}{1+x^2}. \end{aligned} \quad \texttt{(3 points)}$$

One finds

$$u_1' = -\frac{x}{1+x^2}, \quad u_2' = \frac{1}{1+x^2},$$

and one can take

$$u_1 = -\frac{1}{2}\ln(1+x^2), \quad u_2 = \arctan x. \quad \texttt{(3 points)}$$

The general solution is then

$$y = e^x\left(c_1 - \frac{1}{2}\ln(1+x^2)\right) + xe^x(c_2 + \arctan x). \quad \texttt{(2 points)}$$

6. (a) (8%) Solve $x\dfrac{dy}{dx} = \sqrt{1+y^2}$, $y(1) = 1$ for $x > 0$.

(b) (8%) Solve $(1+x^2)\dfrac{dy}{dx} + y = e^{\tan^{-1} x}$, $y(0) = 0$.

Solution:

(a)

$$\frac{dy}{\sqrt{1+y^2}} = \frac{dx}{x} \text{ and } \int \frac{dy}{\sqrt{1+y^2}} = \int \frac{dx}{x} \qquad \text{(寫出其中一種給2分)}$$

$$\ln|y+\sqrt{1+y^2}| = \ln(y+\sqrt{1+y^2}) = \ln x + c \qquad \text{(積分各自2分，共4分)}$$

$$\therefore y + \sqrt{1+y^2} = Ax$$

$$x = 1, y = 1 \therefore (1+\sqrt{2}) = A \qquad \text{(}A\text{ 算對，給2分)}$$

$$y + \sqrt{1+y^2} = (1+\sqrt{2})x$$

(b)

$$\frac{dy}{dx} + \frac{y}{1+x^2} = \frac{1}{1+x^2}e^{\tan^{-1} x}$$

$$\text{Integrating factor } I = e^{\int \frac{dx}{1+x^2}} = e^{\tan^{-1} x} \qquad \text{(2分)}$$

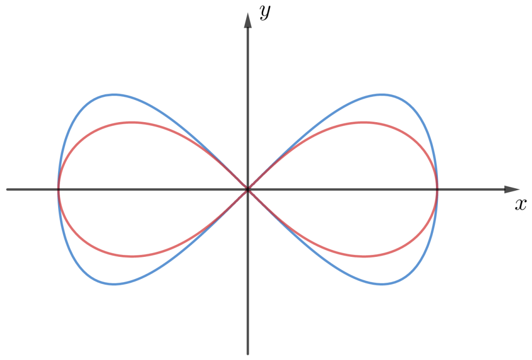
$$(^{\tan^{-1} x} \cdot y)' = \frac{1}{1+x^2}e^{2\tan^{-1} x} \qquad \text{(寫出此式，2分)}$$

$$\therefore y = e^{-\tan^{-1} x}\left\{\frac{1}{2}e^{2\tan^{-1} x} + c\right\} \qquad \text{(積分算出來，2分)}$$

$$x = 0, y = 0, 0 = \frac{1}{2} + c, c = -\frac{1}{2} \qquad \text{算出 } c \text{ 得2分}$$

$$y = \frac{1}{2}e^{\tan^{-1} x} - \frac{1}{2}e^{-\tan^{-1} x}.$$

7. Curve C_1 is defined by the equation $y^2 = x^2(1 - x^2)$. Curve C_2 satisfies the polar equation $r^2 = \cos(2\theta)$.



(a) (3%) Derive a polar equation for C_1. Determine which curve is the outer curve.

(b) (4%) Find tangent lines of C_1 and C_2 at the origin.

(c) (5%) Find the area of the region between two curves.

Solution:

(a) Setting $x = r\cos\theta$ and $y = r\sin\theta$ in the equation $y^2 = x^2(1-x^2)$ of C_1 and removing the factor r^2, one reaches the equation

$$\tan^2\theta = 1 - r^2\cos^2\theta,$$

i. e., the polar equation

$$r^2 = \cos 2\theta/\cos^4\theta. \tag{1}$$

Since $r^2 \geqslant 0$, $\cos 2\theta \geqslant 0$, and hence $\theta \in [-\frac{\pi}{4}, \frac{\pi}{4}]$. For any $\theta \in [-\frac{\pi}{4}, \frac{\pi}{4}]$, it is clear that $\cos 2\theta/\cos^4\theta \geqslant \cos 2\theta$, and hence C_1 is the outer curve.

(b) One can use the parametrization $(x(\theta), y(\theta))$ to compute $(x'(\frac{\pi}{4}), y'(\frac{\pi}{4}))$ and $(x'(-\frac{\pi}{4}), y'(-\frac{\pi}{4}))$. A more reasonable way is to find all possible θ_0 such that $r(\theta_0) = 0$ (which are $\frac{\pi}{4}$ and $-\frac{\pi}{4}$ in the current case) and note that the slope of the tangent at the origin associated to the angle θ_0 is exactly

$$\lim_{\theta\to\theta_0^\pm} \frac{r(\theta)\sin\theta - 0}{r(\theta)\cos\theta - 0} = \lim_{\theta\to\theta_0^\pm} \tan\theta = \tan(\theta_0^\pm),$$

where $\pm$ depends on the question; for example, in our question the answer will be $\tan(\frac{\pi}{4})^- = 1$ and $\tan(-\frac{\pi}{4})^+ = -1$.

(c) Let $r_1(\theta)^2 = \cos 2\theta/\cos^4\theta$ and $r_2(\theta)^2 = \cos 2\theta$. By symmetry, the area between the curves is

$$\begin{aligned}
&4\int_0^{\frac{\pi}{4}} \frac{1}{2} r_2(\theta)^2\, d\theta - 4\int_0^{\frac{\pi}{4}} \frac{1}{2} r_1(\theta)^2\, d\theta \\
&= 2\int_0^{\frac{\pi}{4}} (\cos^2\theta - \sin^2\theta)\frac{1}{\cos^4\theta} d\theta - 2\int_0^{\frac{\pi}{4}} \cos 2\theta\, d\theta \\
&= 2\int_0^{\frac{\pi}{4}} (1 - \tan^2\theta)\sec^2\theta d\theta - 2\int_0^{\frac{\pi}{4}} \cos 2\theta\, d\theta \\
&= 2\int_0^1 (1 - t^2) dt - \sin 2\theta\Big|_0^{\frac{\pi}{4}} = 1/3.
\end{aligned}$$

8. (01-02班) Note that the function $f(t) = \begin{cases} 1, & \text{if } t = 0 \\ \dfrac{\sin t}{t}, & \text{if } t > 0 \end{cases}$ satisfies the inequality $|f(t)| \leqslant 1$, and hence is of exponential order c for every $c > 0$. In particular, you may freely use the properties of the Laplace transform of f introduced in the lectures. Now consider the Laplace transform

$$F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty f(t)e^{-st}dt \quad (s > 0).$$

(a) (4%) Show that $\lim_{s\to\infty} F(s) = 0$. Do NOT quote directly the fact that if f is of exponential order then $\lim_{s\to\infty} \mathcal{L}\{f(t)\}(s) = 0$. (**Hint.** Note that $|\sin t| \leqslant |t|$. Make a comparison of $F(s)$ with a suitable integral, and then apply the squeeze theorem.)

(b) (8%) Find $\lim_{s\to 0^+} F(s)$. (**Hint.** Compute $F'(s)$ and use the fundamental theorem of calculus. You may use the fact that $\mathcal{L}\{tf(t)\} = -F'(s)$.)

Solution:

(a) Since $-1 \leqslant f \leqslant 1$, we have

$$-\frac{1}{s} = -\int_0^\infty e^{-st}dt \leqslant F(s) \leqslant \int_0^\infty e^{-st}dt = \frac{1}{s},$$

and hence $\lim_{s\to\infty} F(s) = 0$.

(b) By the differentiation property of Laplace transformation, we have

$$F'(s) = \int_0^\infty (-t)f(t)e^{-st}dt = -\int_0^\infty \sin t e^{-st}dt = -\mathcal{L}\{\sin t\}(s) = -\frac{1}{s^2+1},$$

and hence

$$F(s) = -\arctan s + C \quad (s > 0).$$

By (1) we know that $\lim_{s\to 0^+} F(s) = C = \lim_{s\to\infty} \arctan s = \frac{\pi}{2}$.