1. (22%) 求下列極限。(不能使用 l'Hospital's Rule)

(a) (4%) $\lim_{x\to-\infty} \frac{e^{-x}+2}{3e^{x}-e^{-x}}$.

(b) (6%) $\lim_{x\to\infty} \sqrt{x^2-x}-x+\tan^{-1} x$.

(c) (6%) $\lim_{x\to 0} \frac{\cos x - 1}{x^2}$.

(d) (6%) $\lim_{x\to 1} \frac{\ln x}{x^3-1}$.

Solution:

(a)

$$\lim_{x\to-\infty} \frac{e^{-x}+2}{3\,e^{x}-e^{-x}} = \lim_{x\to-\infty} \frac{1+2\,e^{x}}{3\,e^{2\,x}-1} \qquad \text{multiply } e^{x} \text{ (2pts)}$$

$$= -1 \qquad \lim_{x\to-\infty} e^{x} = \lim_{x\to-\infty} e^{2\,x} = 0 \text{ (2pts)}$$

(b)

$$\lim_{x\to\infty} \sqrt{x^2-x}-x+\tan^{-1} x$$

$$= \left(\lim_{x\to\infty} \sqrt{x^2-x}-x\right) + \left(\lim_{x\to\infty} \tan^{-1} x\right) \qquad \text{limit law (1pt)}$$

$$= \left(\lim_{x\to\infty} \sqrt{x^2-x}-x\right) + \frac{\pi}{2} \qquad \lim_{x\to\infty} \tan^{-1} x = \frac{\pi}{2} \text{ (1pts)}$$

$$= \lim_{x\to\infty} \frac{(\sqrt{x^2-x}-x)(\sqrt{x^2-x}+x)}{(\sqrt{x^2-x}+x)} + \frac{\pi}{2} \qquad \text{multiply } \sqrt{x^2-x}+x \text{ (1pt)}$$

$$= \lim_{x\to\infty} \frac{-x}{\sqrt{x^2-x}+x} + \frac{\pi}{2} \qquad \text{simplify using } a^2-b^2 = (a+b)\,(a-b) \text{ (1pt)}$$

$$= \lim_{x\to\infty} \frac{-1}{\sqrt{1-\frac{1}{x}}+1} + \frac{\pi}{2} \qquad \text{divided by } x \text{ (1pt)}$$

$$= -\frac{1}{2} + \frac{\pi}{2} \qquad \lim_{x\to\infty} \frac{1}{x} = 0 \text{ (1pt)}$$

(c)

$$\lim_{x\to 0} \frac{\cos x - 1}{x^2}$$

$$= \lim_{x\to 0} \frac{(\cos x - 1)(\cos x + 1)}{x^2\,(\cos x + 1)} \qquad \text{multiply } (\cos x + 1) \text{ (2pts)}$$

$$= \lim_{x\to 0} \frac{-\sin^2 x}{x^2\,(\cos x + 1)} \qquad \cos^2 x - 1 = -\sin^2 x \text{ (1pt)}$$

$$= -\left(\lim_{x\to 0} \frac{\sin^2 x}{x^2}\right)\left(\lim_{x\to 0} \frac{1}{\cos x + 1}\right) \qquad \text{limit law (1pt)}$$

$$= -1 \cdot \frac{1}{2} = -\frac{1}{2} \qquad \lim_{x\to 0} \frac{\sin x}{x} = 1 \text{ (2pts)}$$

(d)

$$\lim_{x\to 1} \frac{\ln x}{x^3-1}$$

$$= \lim_{x\to 1} \frac{\ln x}{(x-1)\,(x^2+x+1)} \qquad x^3-1 = (x-1)\,(x^2+x+1) \text{ (2pts)}$$

$$= \left(\lim_{x\to 1} \frac{\ln x}{x-1}\right)\left(\lim_{x\to 1} \frac{1}{x^2+x+1}\right) \qquad \text{limit law (1pt)}$$

$$= 1 \cdot \frac{1}{3} = \frac{1}{3} \qquad \lim_{x\to 1} \frac{\ln x}{x-1} = 1 \text{ (3pts)}$$

2. (20%) (a) (6%) 令 $f(x) = \dfrac{x}{3x^2+4}$，求 $f'(x)$.

(b) (7%) 令 $f(x) = \tan^{-1}(e^x) + \sin^{-1}(\cos(2x))$，求 $f'(x)$.

(c) (7%) 令 $f(x) = x^{\sin x}$，求 $f'(\pi)$.

Solution:

(a) $f(x) = \frac{x}{3x^2+4}$. Then

$$f'(x) = \frac{3x^2+4-x\cdot 6x}{(3x^2+4)^2} \quad (4\%)$$
$$= \frac{-3x^2+4}{(3x^2+4)^2} \quad (2\%).$$

(b) $f(x) = \tan^{-1}(e^x) + \sin^{-1}(\cos(2x))$. Then

$$f'(x) = \frac{1}{1+e^{2x}}e^x + \frac{1}{\sqrt{1-\cos(2x)^2}}(-\sin(2x)\cdot 2)$$
$$= \frac{e^x}{1+e^2x} - \frac{2\sin(2x)}{|\sin(2x)|}.$$

第一個等號分成四部分，每一部分兩分，全對給七分。 可不化簡為第二個等號，但是化簡錯扣一分。

(c) $f(x) = x^{\sin x}$. Then

$$\ln f(x) = \sin x \ln x. \quad (2\%)$$

We have

$$\frac{f'(x)}{f(x)} = \cos x \ln x + \frac{\sin x}{x}. \quad (3\%)$$

Thus

$$f'(\pi) = -\ln \pi. \quad (2\%)$$

或者(另法)

$$f(x) = e^{\sin x \ln x} \quad (2\%).$$

Then

$$f'(x) = x^{\sin x}\left(\cos x \cdot \ln x + \frac{\sin x}{x}\right) \quad (3\%).$$

Thus

$$f'(\pi) = -\ln \pi. \quad (2\%)$$

3. (10%) 考慮方程式 $x^3 + xy + \frac{y^3}{8} = 4$ 決定的平面曲線。

(a) (6%) 求在點 $(1, 2)$ 的 $\frac{dy}{dx}$。

(b) (4%) 曲線在點 $(1, 2)$ 附近可描寫為 $y = f(x)$。用線性逼近估計 $f(1.01)$。

Solution:

(a)

$$3x^2 + y + x\frac{dy}{dx} + \frac{3}{8}y^2\frac{dy}{dx} = 0$$ implicit differentiation (3pts)

$$\frac{dy}{dx} = -\frac{24x^2 + 8y}{8x + 3y^2}$$ rearrange and find $\frac{dy}{dx}$ (1pt)

$$\left.\frac{dy}{dx}\right|_{(x,y)=(1,2)} = -\frac{24 + 16}{8 + 12}$$ put $(x, y) = (1, 2)$ (1pt)

$$\left.\frac{dy}{dx}\right|_{(x,y)=(1,2)} = -2$$ simplify (1pt)

(b)

$$f(1.01) \approx f(1) + f'(1)(1.01 - 1)$$ $f(x) \approx f(a) + f'(a)(x - a)$ (2pts)

with $a = 1$ and $x = 1.01$

$$= 2 + (-2)(1.01 - 1)$$ $f(1) = 2$ and $f'(1) = -2$ (by (a)) (1pt)

$$= 1.98.$$ simplify (1pt)

4. (16%) 令 $f(x) = \tan x$, $x \in [0, \frac{\pi}{2})$.

   (a) (6%) 證明當 $x \in (0, \frac{\pi}{2})$, $\tan x > x$.

   (b) (4%) 由(a)解釋 $f'(x) = 1 + f^2(x) > 1 + x^2$，其中 $x \in (0, \frac{\pi}{2})$。

   (c) (6%) 證明當 $x \in (0, \frac{\pi}{2})$, $\tan x > x + \frac{1}{3}x^3$。

Solution:

(a) By MVT,

$$\begin{aligned}\frac{\tan x - \tan 0}{x - 0} &= \sec^2(c), \quad (3\%)\\ &> 1 \quad \text{since } c \in (0, \pi/2) \quad (2\%)\end{aligned}$$

Since $x > 0$ (1%), we have $\tan x > x$.

或者(另法)

Let $f(x) = \tan x - x$. Then

$$\begin{aligned}f'(x) &= \sec^2 x - 1 \quad (3\%)\\ &> 0 \quad \text{on } (0, \pi/2). \quad (2\%)\end{aligned}$$

For $x \in (0, \pi/2)$, we have

$$f(x) > f(0) = 0 \Rightarrow \tan x > x. \quad (1\%)$$

(b) $f(x) = \tan x$.

$$\begin{aligned}f'(x) &= \sec^2 x \quad (1\%)\\ &= 1 + \tan^2 x = 1 + f^2(x) \quad (2\%)\\ &> 1 + x^2 \quad \text{by } (a) \quad (1\%).\end{aligned}$$

(c) Let $G(x) = \tan x - (x + x^3/3)$. Then

$$\begin{aligned}G'(x) &= \sec^2 x - (1 + x^2) \quad (2\%)\\ &> 0 \quad \text{by } (b) \quad (2\%).\end{aligned}$$

For $x \in (0, \pi/2)$, we have

$$G(x) > G(0) = 0 \Rightarrow \tan x > x + \frac{x^3}{3}. \quad (2\%).$$

5. (20%) 畫函數 $f(x) = \ln|x+1| + 5x + x^2$ 的圖形。

(a) (2%) 寫出 $f(x)$ 的定義域。求 $\lim_{x\to -1^-} f(x)$ 和 $\lim_{x\to -1^+} f(x)$。

(b) (7%) 求 $f'(x)$。找出 $f(x)$ 遞增、遞減的區間和 $f(x)$ 的局部極值。

(c) (7%) 求 $f''(x)$。判斷 $y = f(x)$ 的凹性，並求反曲點。

(d) (4%) 畫函數圖形 $y = f(x)$.

Solution:

(a) The domain of $f(x)$ is $\mathbb{R} \setminus \{-1\}$ (1 pt)
because $\lim_{x\to -1^-} \ln|x+1| = -\infty$ and $\lim_{x\to -1^-} 5x + x^2 = -4$
$\therefore \lim_{x\to -1^-} f(x) = \lim_{x\to -1^-} \ln|x+1| + 5x + x^2 = -\infty$ (1 pt, Student can write down these limits directly without reasoning.)
Similarly $\lim_{x\to -1^+} f(x) = -\infty$

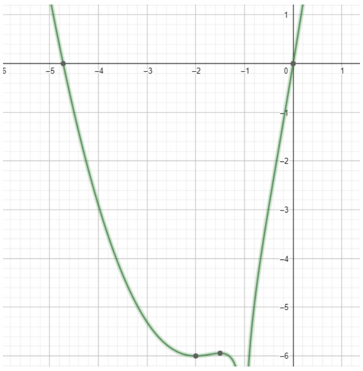
(b)

$$
\begin{aligned}
f'(x) &= \frac{1}{x+1} + 5 + 2x && \text{(2 pts)} \\
&= \frac{(2x+3)(x+2)}{(x+1)} && \text{(1 pt for factorization)}
\end{aligned}
$$

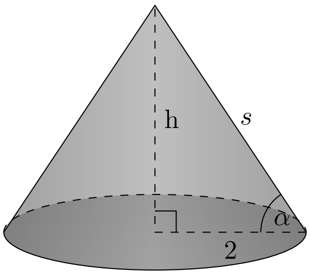
$f'(x) > 0$ for $x \in (-2, -\frac{3}{2}) \cup (-1, \infty)$. Hence $f(x)$ is increasing on $(-2, -\frac{3}{2})$ and $(-1, \infty)$. (1 pt)
$f'(x) < 0$ for $x \in (-\infty, -2) \cup (-\frac{3}{2}, -1)$. Hence $f(x)$ is decreasing on $(-\infty, -2)$ and $(-\frac{3}{2}, -1)$ (1 pt)
$f(-2)$ is a local minimum. (1 pt)
$f(-\frac{3}{2})$ is a local maximum. (1 pt)

(c) $f''(x) = \frac{-1}{(x+1)^2} + 2 = 2\frac{(x+1-\frac{1}{\sqrt{2}})(x+1+\frac{1}{\sqrt{2}})}{(x+1)^2}$ (3 pts)
$f''(x) > 0$ for $x \in (-\infty, -1-\frac{1}{\sqrt{2}}) \cup (-1+\frac{1}{\sqrt{2}}, \infty)$.
Hence $y = f(x)$ is concave upward on $(-\infty, -1-\frac{1}{\sqrt{2}})$ and $(-1+\frac{1}{\sqrt{2}}, \infty)$. (1 pt)
$(-1-\frac{1}{\sqrt{2}}, f(-1-\frac{1}{\sqrt{2}}))$ and $(-1+\frac{1}{\sqrt{2}}, f(-1+\frac{1}{\sqrt{2}}))$ are inflection points (2 pts)

(d) $f(-2) = -6$ $f(-\frac{3}{2}) = -\ln 2 - \frac{21}{4}$
$f(-1-\frac{1}{\sqrt{2}}) = -\frac{1}{2}\ln 2 - \frac{7}{2} - \frac{3}{\sqrt{2}}$, $f(-1+\frac{1}{\sqrt{2}}) = -\frac{1}{2}\ln 2 - \frac{7}{2} - \frac{3}{\sqrt{2}}$
(1 pt for the approximating values of $f(-2)$, $f(-\frac{3}{2})$, $f(-1-\frac{1}{\sqrt{2}})$, $f(-1+\frac{1}{\sqrt{2}})$)
The graph of $f(x)$ on $(-\infty, -2)$, $(-2, -1-\frac{1}{\sqrt{2}})$, $(-1-\frac{1}{\sqrt{2}}, -\frac{3}{2})$, $(-\frac{3}{2}, -1)$, $(-1, -1+\frac{1}{\sqrt{2}})$, $(-1+\frac{1}{\sqrt{2}}, \infty)$ each counts for $\frac{1}{2}$ pt.



6. (12%) 一盞燈被掛在一半徑是 2m 的圓桌圓心正上方。求燈離桌面的高度 h，使得圓桌圓周上的照明度 I 最大。已知 $I = \frac{k\sin\alpha}{s^2}$，其中 $k > 0$ 是常數，s 是斜邊長，α 是光線照到圓周上的角度。(需要說明求出來的極值是最大值。)



Solution:

$s = \sqrt{h^2+4}$. $\sin\alpha = \frac{h}{s} = \frac{h}{\sqrt{h^2+4}}$.

Hence $I(h) = \frac{k\sin\alpha}{s^2} = k\frac{h}{(h^2+4)^{3/2}}$ for $h \in (0,\infty)$ or $h \in [0,\infty)$

(4 pts for correct function $I(h)$. 1 pt for correct domain $h \in (0,\infty)$ or $[0,\infty)$.)

$I'(h) = k\frac{4-2h^2}{(h^2+4)^{5/2}}$ (3 pts)

$I'(h) = 0 \Rightarrow h = \pm\sqrt{2}$ (1 pt)

$\because I'(h) > 0$ for $h \in (0,\sqrt{2})$ and $I'(h) < 0$ for $h \in (\sqrt{2},\infty)$

$\therefore I(\sqrt{2})$ is the absolute maximum of $I(h)$ on $(0,\infty)$. (3 pts)

Another reasoning:

$\because I(0) = 0$, $\lim_{h\to\infty} I(h) = 0$, and $I(\sqrt{2}) = \frac{k}{6\sqrt{3}}$

$\therefore I(2) > I(0)$, $I(\sqrt{2}) > \lim_{h\to\infty} I(h)$.

And we can conclude that $I(\sqrt{2})$ is the absolute maximum of $I(h)$ on $(0,\infty)$.