# 1091 微乙01-03班 期末考解答和評分標準

1. Compute the integrals.

(a) (6%) $\int x(\sin 2x + \cos x)dx$.

(b) (9%) $\int \dfrac{8x-4}{x^2(x^2+4)}dx$.

**Solution:**

(a)

$$\int x\,(\sin(2\,x)+\cos x)\,dx$$

$$=\int x\,\sin(2\,x)\,dx+\int x\,\cos x\,dx$$

$$=\frac{-1}{2}\int x\,d\cos(2\,x)+\int x\,d\sin x$$

$d\cos(2\,x)=-2\,\sin(2\,x)\,dx$ (1pt)

$d\sin x=\cos x\,dx$ (1pt)

$$=\frac{-1}{2}\left(x\,\cos(2\,x)-\int\cos(2\,x)\,dx\right)$$

IBP (2pts)

$$+x\,\sin x-\int\sin x\,dx$$

$\int\cos(2\,x)\,dx=\frac{1}{2}\,\sin(2\,x)+C$ (0.5pt)

$$=\frac{-1}{2}\left(x\,\cos(2\,x)-\frac{1}{2}\,\sin(2\,x)\,dx\right)$$

$\int\sin x\,dx=-\cos x+C$ (0.5pt)

$$+x\,\sin x+\cos x+C$$

C (1pt)

(b)

$$\frac{8x-4}{x^2\,(x^2+4)}$$

$$=\frac{A+B\,x}{x^2+4}+\frac{C}{x^2}+\frac{D}{x}$$

partial fractions

$$=\frac{1-2x}{x^2+4}-\frac{1}{x^2}+\frac{2}{x}$$

Solve $\Rightarrow$ $A=1,\ B=-2,\ C=-1,\ D=2$ (4pts)

$$\int\frac{8x-4}{x^2\,(x^2+4)}\,dx$$

$$=\int\left(\frac{1}{x^2+4}-\frac{2x}{x^2+4}-\frac{1}{x^2}+\frac{2}{x}\right)dx$$

use (a)

$$=\frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right)-\ln\left(x^2+4\right)+\frac{1}{x}+2\ln x+C$$

4 integrals (4pts); C (1pt)

2. (a) (3%) Evaluate and simplify $\dfrac{d}{dx}\ln\left(\sqrt{x^2+1}+x\right)$.

(b) (5%) Evaluate $\displaystyle\int \sec x\,dx$.

(c) (7%) Use (b) and trigonometric substitution to find $\displaystyle\int_0^1 \frac{1}{\sqrt{x^2+1}}\,dx$.

(d) (7%) Use (a) and integration by parts to evaluate the integral $\displaystyle\int_0^1 \ln\left(\sqrt{x^2+1}+x\right)\,dx$.

**Solution:**

(a)

$$\frac{d}{dx}\ln\left(\sqrt{x^2+1}+x\right) = \frac{\frac{1}{2}(x^2+1)^{\frac{-1}{2}}(2x)+1}{\sqrt{x^2+1}+x} \qquad (\ln x)' = \frac{1}{x} \text{ (1pt) and chain rule (1pt)}$$

$$= \frac{\sqrt{x^2+1}+x}{(\sqrt{x^2+1}+x)(\sqrt{x^2+1})} \qquad \text{multiply } (x^2+1)^{\frac{1}{2}} \text{ (1pt)}$$

$$= \frac{1}{\sqrt{x^2+1}}$$

(b)

$$\int \sec x\,dx = \int \frac{\sec x(\sec x+\tan x)}{\sec x+\tan x}\,dx \qquad \text{multiply } (\sec x+\tan x) \text{ (2pts)}$$

$$= \int \frac{d(\sec x+\tan x)}{\sec x+\tan x} \qquad \frac{d}{dx}\sec x = \sec x\tan x \text{ (1pt)}, \ \frac{d}{dx}\tan x = \sec^2 x \text{ (1pt)}$$

$$= \ln|\sec x+\tan x| + C \qquad \int \frac{1}{x}\,dx = \ln|x| + C \text{ (1pt)}$$

(c)

$$\int_0^1 \frac{1}{\sqrt{x^2+1}}\,dx \qquad x := \tan\theta \Rightarrow dx = \sec^2\theta\,d\theta \text{ (2pt)}$$

$$x = 1 \Rightarrow \theta = \pi/4 \text{ (0.5pt)}, \ x = 0 \Rightarrow \theta = 0 \text{ (0.5pt)}$$

$$= \int_0^{\pi/4} \frac{1}{\sec\theta}\sec^2\theta\,d\theta \qquad \text{simplify (1pt)}$$

$$= \int_0^{\pi/4} \sec\theta\,d\theta \qquad \text{use (b) (1pt)}$$

$$= \ln|\sec\theta+\tan\theta|\Big|_0^{\pi/4} \qquad \text{simplify (1pt)}$$

$$= \ln|\sec(\pi/4)+\tan(\pi/4)| \qquad \sec(\pi/4) = \sqrt{2} \text{ (0.5pt)}; \ \tan(\pi/4) = 1 \text{ (0.5pt)}$$

$$= \ln(\sqrt{2}+1)$$

(d)

$$\int_0^1 \ln\left(\sqrt{x^2+1}+x\right)\,dx$$

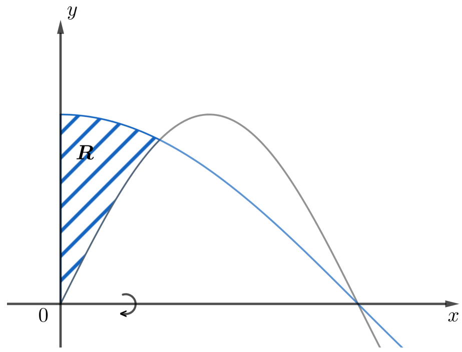
$$= x\,\ln\left(\sqrt{x^2+1}+x\right)\Big|_0^1 - \int_0^1 x\,d\ln\left(\sqrt{x^2+1}+x\right)$$ IBP: $u = \ln\left(\sqrt{x^2+1}+x\right)$, $v = x$ (2pts)

$$= \ln(\sqrt{2}+1) - \int_0^1 x\,\frac{dx}{\sqrt{x^2+1}}$$ evaluate bdy. term (1pt); use (a) (1pt)

$$= \ln(\sqrt{2}+1) - (x^2+1)^{\frac{1}{2}}\Big|_0^1$$ $\int \frac{x\,dx}{\sqrt{x^2+1}} = (x^2+1)^{\frac{1}{2}}$ (2pts)

$$= \ln(\sqrt{2}+1) - \sqrt{2} + 1$$ evaluate bdy. term (1pt)

3. (a) (10%) Let $R$ be the region bounded by $y = \cos x$, $y = \sin 2x$ and $x = 0$ in the first quadrant. Rotate $R$ about the $x$-axis. Find the volume of the resulting solid.




**Solution:**

First, find intersection points of $y = \cos x$ and $y = \sin 2x$ for $0 \le x \le \frac{\pi}{2}$.

$y = \cos x = \sin 2x = 2\sin x \cos x \Rightarrow \sin x = \frac{1}{2}$ or $\cos x = 0$ i.e. $x = \frac{\pi}{6}$ or $\frac{\pi}{2}$.

Hence $R$ is bounded by $y = \cos x$ and $y = \sin 2x$, with $0 \le x \le \frac{\pi}{6}$. (2 pts for the range of $x$.)

By the disc method, the volume is

$$V = \pi \int_0^{\frac{\pi}{6}} (\cos x)^2 - (\sin 2x)^2 dx$$ (2 pts for the formula.

1 pt for recognizing $\cos x \ge \sin 2x$ for $0 \le x \le \frac{\pi}{6}$.)

$$= \pi \int_0^{\frac{\pi}{6}} \frac{1+\cos 2x}{2} - \frac{1-\cos 4x}{2} dx$$ (2 pts for half angle formulas)

$$= \frac{\pi}{2}\left[\frac{\sin 2x}{2} + \frac{\sin 4x}{4}\right]\Big|_{x=0}^{x=\frac{\pi}{6}}$$ (2 pts for integrating $\cos 2x$ and $\cos 4x$.)

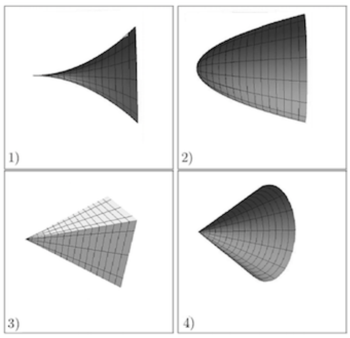
$$= \frac{3\sqrt{3}}{16}\pi$$ (1 pt for the final answer.)

If Students write $V = \int_0^{\frac{\pi}{6}} 2\pi x(\cos x - \sin 2x)dx = \cdots$, they get 2 pts for the range of $x$ $(0 \le x \le \frac{\pi}{6})$ and 1 pt for recognizing $\cos x \ge \sin 2x$ for $0 \le x \le \frac{\pi}{6}$.

(b) (6%) (A) $\int_0^1 \pi x \, dx$. (B) $\int_0^1 \pi x^4 \, dx$. (C) $\int_0^1 \pi x^2 \, dx$. (D) $\int_0^1 x^2 \, dx$.
Match each solid with the integral that represents its volume.

| Integrals | A | B | C | D |
|---|---|---|---|---|
| Solid | 2 | 1 | 4 | 3 |

1.5 pts for each answers.



4. (8%) Find the length of the curve

$$y = f(x) = \int_1^x \sqrt{t^3 - 1}\, dt, \quad 1 \le x \le 4.$$

**Solution:**

By the fundamental theorem of calculus

$$f'(x) = \sqrt{x^3 - 1}. \quad (2\%)$$

$$\begin{aligned} \text{length} &= \int_1^4 \sqrt{1 + f'(x)^2}\, dx \quad (2\%) \\ &= \int_1^4 x^{3/2}\, dx \quad (2\%) \\ &= \frac{62}{5} \quad (2\%). \end{aligned}$$

5. Let $f(x) = xe^x$. (When finding the following Taylor series, you don't need to specify the range of $x$ for which the series equals the function.)

(a) (4%) Find the Taylor series for $f(x)$ at $x = 0$.

(b) (7%) Calculate $\int_0^x te^t\,dt$ and find its Taylor series at $x = 0$.

(c) (4%) Find the sum $\sum_{n=0}^{\infty} \frac{1}{n!(n+2)}$.

**Solution:**

(a)

$$\begin{aligned} f(x) = xe^x &= x\left(1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots\right) \quad (2\%) \\ &= x + \frac{x^2}{1!} + \frac{x^3}{2!} + \cdots + \frac{x^{n+1}}{n!} + \cdots \quad (2\%). \end{aligned}$$

(b)

$$\begin{aligned} \int_0^x te^t\ dt &= \int_0^x t\,de^t \quad (2\%) \\ &= xe^x - e^x + 1. \quad (2\%) \end{aligned}$$

Use term-be-term integration for the Taylor expansion of $te^t$ in (a), we have

$$\begin{aligned} \int_0^x te^t\ dt &= \int_0^x t\,dt + \int_0^x \frac{t^2}{1!}\,dt + \int_0^x \frac{t^3}{2!}\,dx + \cdots + \int_0^x \frac{t^{n+1}}{n!}\,dt + \cdots \quad (1\%) \\ &= \frac{x^2}{2} + \frac{x^3}{3\cdot 1!} + \cdots + \frac{x^{n+2}}{(n+2)\cdot n!} + \cdots. \quad (2\%) \end{aligned}$$

(c) By (b), we have

$$xe^x - e^x + 1 = \frac{x^2}{2} + \frac{x^3}{3\cdot 1!} + \cdots + \frac{x^{n+2}}{(n+2)\cdot n!} + \cdots. \quad (1\%)$$

Put $x = 1$ (1%), we have

$$1 = \frac{1}{2} + \frac{1}{3\cdot 1!} + \frac{1}{4\cdot 2!} + \cdots + \frac{1}{(n+2)\cdot n!} + \cdots. \quad (2\%)$$

6. (a) (6%) Find the Taylor series for $f(x) = \ln(1-x^2)$, $g(x) = \cos x - 1$, and $h(x) = \sin(2x^4)$ at $x = 0$. (You don't need to specify the range of $x$ for which the function equals its Taylor series.)

(b) (4%) Evaluate $\lim_{x\to 0} \frac{(\cos x - 1)\ln(1-x^2)}{\sin(2x^4)}$.

**Solution:**

(a) $\ln(1-y) = -y - \frac{y^2}{2} - \frac{y^3}{3} - \cdots - \frac{y^{n+1}}{n+1} - \cdots$, for $-1 \le y < 1$.

$f(x) = \ln(1-x^2) = -x^2 - \frac{x^4}{2} - \frac{x^6}{3} - \cdots - \frac{x^{2x+2}}{n+1} - \cdots$, for $-1 < x < 1$

(1 pt for knowing the Taylor series for $\ln(1-y)$. 1 pt for the Taylor series for $f(x)$.)

$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots + \frac{(-1)^n x^{2n}}{(2n)!} + \cdots$, for $x \in \mathbb{R}$

$g(x) = \cos x - 1 = -\frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots + \frac{(-1)^n x^{2n}}{(2n)!} + \cdots$ for $x \in \mathbb{R}$

(1 pt for knowing the Taylor series for $\cos x$. 1 pt for the Taylor series for $g(x)$.)

$\sin y = y - \frac{y^3}{3!} + \frac{y^5}{5!} + \cdots + \frac{(-1)^n}{(2n+1)!} y^{2n+1} + \cdots$

$h(x) = \sin(2x^4) = 2x^4 - \frac{2^3 x^{12}}{3!} + \cdots + \frac{(-1)^n 2^{2n+1}}{(2n+1)!} x^{8n+4} + \cdots$

(1 pt for knowing the Taylor series for $\sin y$. 1 pt for the Taylor series for $h(x)$.)

(b) **Solution 1:**

$$\frac{(\cos x - 1)\ln(1-x^2)}{\sin(2x^4)} = \frac{(-\frac{x^2}{2} + \frac{x^4}{4!} - \cdots)(-x^2 - \frac{x^4}{2} - \cdots)}{2x^4 - \frac{4}{3}x^{12} + \cdots}$$ (2 pts for plugging in Taylor series)

$$= \frac{\frac{1}{2}x^4 + (\frac{1}{4} - \frac{1}{4!})x^6 \cdots}{2x^4 - \frac{4}{3}x^{12} + \cdots} \to \frac{1}{4} \text{ as } x \to 0$$

Hence $\lim_{x\to 0} \frac{(\cos x - 1)\ln(1-x^2)}{\sin(2x^4)} = \frac{1}{4}$ (2 pts for final answers)

**Solution 2:**

$$\lim_{x\to 0} \frac{(\cos x - 1)\ln(1-x^2)}{\sin(2x^4)} \overset{\frac{0}{0}}{\underset{\text{L'H}}{=\!=}} \lim_{x\to 0} \frac{-\sin x \cdot \ln(1-x^2) + (\cos x - 1)\frac{-2x}{1-x^2}}{\cos(2x^4)\cdot 8x^3}$$

$$= \lim_{x\to 0} -\left(\frac{\sin x}{x}\frac{\ln(1-x^2)}{x^2}\frac{1}{8\cos(2x^4)}\right) + \left(\frac{\cos x - 1}{x^2}\frac{-1}{4(1-x^2)\cos(2x^4)}\right)$$

$$= \frac{1}{8} + \frac{1}{8} = \frac{1}{4}.$$

(2 pts for using L'Hospital's Rule and obtaining correct first derivatives. 2 pts for final answers.)

7. (a) (6%) Compute $\lim_{x\to 0} \frac{\int_0^{4x^2} \cos(\sqrt{t})dt}{x^2}$ by L'Hospital's Rule.

(b) (8%) Compute $\lim_{x\to\infty} (1+x)^{1/\ln x}$.

**Solution:**

(a) As $x \to 0$, $\int_0^{4x^2} \cos(\sqrt{t})dt \to 0$ and $x^2 \to 0$.
Hence we can apply L'Hospital's rule.

$$\lim_{x\to 0} \frac{\int_0^{4x^2} \cos(\sqrt{t})dt}{x^2} \overset{\frac{0}{0}}{\underset{\text{L'H}}{=\!=}} \lim_{x\to 0} \frac{\cos(2|x|)\cdot 8x}{2x} = 4$$

(1 pt for using L'H rule. 3 pts for $\frac{d}{dx}\int_0^{4x^2} \cos(\sqrt{t})dt$)
(2 pts for the final answer.)

(b)

$$\ln(1+x)^{\frac{1}{\ln x}} = \frac{1}{\ln x}\cdot \ln(1+x) \qquad \text{(2 pts)}$$

$$\lim_{x\to\infty} \frac{\ln(1+x)}{\ln x} \overset{\frac{\infty}{\infty}}{\underset{\text{L'H}}{=\!=}} \lim_{x\to\infty} \frac{\frac{1}{1+x}}{\frac{1}{x}} \qquad \text{(2 pts)}$$

$$= \lim_{x\to\infty} \frac{x}{1+x} = 1 \qquad \text{(2 pts)}$$

$$\text{Hence } \lim_{x\to\infty} (1+x)^{\frac{1}{\ln x}} = e^1 = e \qquad \text{(2 pts)}$$