1082 微積分**4-**在經濟商管的應用 期考詳解及評分標準

1. (10%) Read the following statements. Answer "**True**" if it is correct. Answer "**False**" if it is incorrect.

   (a) (2%) **True** A is an $n \times m$ matrix. The dimension of A's column space equals the dimension of A's row space.

   (b) (2%) **True** A is an $n \times n$ matrix. If $\det A = 0$, then 0 is an eigenvalue of A.

   (c) (2%) **True** A is an $n \times m$ matrix. The rank of A is less than or equal to n. The rank of A is less than or equal to m.

   (d) (2%) **False** The set $S = \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} -1 \\ 3 \\ -2 \end{bmatrix}, \begin{bmatrix} 1 \\ 7 \\ 4 \end{bmatrix} \right\}$ is a basis for $\mathbb{R}^3$.

   (e) (2%) **False** B is a $k \times n$ matrix. Suppose that there is a $\mathbf{x_0} \neq \mathbf{0}$, $\mathbf{x_0} \in \mathbb{R}^n$ such that $B\mathbf{x_0} = \mathbf{0}$. Then B is not full rank.

2. (18%) Consider the quadratic form $f(x,y,z) = -2x^2 + 2xy - 2y^2 - 5z^2$.

(a) (3%) Express f in the form of

$$f(x,y,z) = (x,y,z)\, M \begin{pmatrix} x \\ y \\ z \end{pmatrix},$$

where M is a real-valued symmetric 3×3 matrix. Find M.

(b) (6%) Find the eigenvalues of M and their corresponding eigenvectors.

(c) (4%) Diagonalize M. That is, find a nonsingular matrix P and a diagonal matrix D such that $P^{-1}MP = D$.

(d) (5%) Use Sylvester's criterion to determine the definiteness of M, and thus f.

Solution:

(a) $M = \begin{pmatrix} -2 & 1 & 0 \\ 1 & -2 & 0 \\ 0 & 0 & -5 \end{pmatrix}$. (3pts)

(b)

$$\begin{aligned} 0 =& \det \begin{pmatrix} -2-\lambda & 1 & 0 \\ 1 & -2-\lambda & 0 \\ 0 & 0 & -5-\lambda \end{pmatrix} && \text{characteristic polynomial} \\ =& (-5-\lambda)\left((-2-\lambda)^2 - 1\right) && \text{expand} \\ =& (-5-\lambda)(\lambda^2 + 4\lambda + 3) && \text{expand} \\ =& -(\lambda+5)(\lambda+1)(\lambda+3) && \text{factor} \\ \lambda =& -5, -1, -3 && \text{solve (3pts)} \end{aligned}$$

(1) $\lambda = -5 \implies \begin{pmatrix} -2+5 & 1 & 0 \\ 1 & -2+5 & 0 \\ 0 & 0 & -5+5 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$\implies$ eigenvector is $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = t\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ for any $t \in \mathbb{R}$ with $t \neq 0$. (1pt)

(2) $\lambda = -3 \implies \begin{pmatrix} -2+3 & 1 & 0 \\ 1 & -2+3 & 0 \\ 0 & 0 & -5+3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$\implies$ eigenvector is $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = s\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$ for any $s \in \mathbb{R}$ with $s \neq 0$. (1pt)

(3) $\lambda = -1 \implies \begin{pmatrix} -2+1 & 1 & 0 \\ 1 & -2+1 & 0 \\ 0 & 0 & -5+1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$\implies$ eigenvector is $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = r\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ for any $r \in \mathbb{R}$ with $r \neq 0$. (1pt)

(c) $P = \begin{pmatrix} 0 & 1 & 1 \\ 0 & -1 & 1 \\ 1 & 0 & 0 \end{pmatrix}$ (2pts) $\implies P^{-1}MP = \begin{pmatrix} -5 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -1 \end{pmatrix} = D$. (2pts)

(d) M is negative definite (2pts), and thus f, since (3pts)

$$-2 < 0,\ \det\begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix} = 3 > 0,\ \det M = -15 < 0.$$

3. (20%) Find the maximum value of the function $f(x,y) = x^2 + 2y^2$ subject to the constraints $x + 2y \le 9$, $x^2 + y^2 \ge 16$, $x \ge 0$ and $y \ge 0$.

(a) (2%) Could both constraints $x + 2y \le 9$ and $x^2 + y^2 \ge 16$ be binding?

(b) (2%) Check whether the Kuhn - Tucker version NDCQ is satisfied.

(c) (2%) Write down the Kuhn - Tucker version Lagrangian function $\tilde{L}$.

(d) (3%) Write down the Kuhn - Tucker version first order conditions.

(e) (4%) Is there a solution such that the constraint $x^2 + y^2 \ge 16$ is binding?

(f) (5%) Find solution(s) such that the constraint $x + 2y \le 9$ is binding.

(g) (2%) Find the maximum value.

Solution:

Let $g_1(x,y) = x + 2y$ and $g_2(x,y) = -x^2 - y^2$. Suppose that (x^*, y^*) is the maximizer of the problem.

(a) We want to find (x,y) satisfies $x + 2y = 9$ and $x^2 + y^2 = 16$. Since $x = 9 - 2y$,

$$(9-2y)^2 + y^2 = 16 \Rightarrow 65 - 36y + 5y^2 = 0 \Rightarrow \text{ it has no real solution becase } 36^2 - 65 \times 20 < 0.$$

(2 points).

(b) We have that $\nabla g_1(x,y) = \langle 1, 2\rangle$ and $\nabla g_2(x,y) = \langle -2x, -2y\rangle$.
From (a), we know that $g_1 \le 9$ and $g_2 \le 16$ can not both binding.
Suppose tht $g_1(x^*, y^*) = 9$. Since $\nabla g_1(x,y) = \langle 1, 2\rangle$, $(\frac{\partial g_1}{\partial x})$, $(\frac{\partial g_1}{\partial y})$, $(\frac{\partial g_1}{\partial x} \quad \frac{\partial g_1}{\partial y})$ all have rank 1. (1 point).
Suppose that $g_2(x^*, y^*) = 16$, then $(x^*)^2 + (y^*)^2 = 16 \Rightarrow x^*$ and y^* can not be both 0. If $x^* > 0$ but $y^* = 0$, then $(\frac{\partial g_2}{\partial x} = (2x^*)$ has rank 1.
If $x^* = 0$ but $y^* > 0$, then $(\frac{\partial g_2}{\partial y} = (2y^*)$ has rank 1.
If $x^* > 0$ and $y^* > 0$, then $(\frac{\partial g_1}{\partial x} \quad \frac{\partial g_1}{\partial y})$ has rank 1. (1 point) Therefore it satisfies Kuhn-Tucker version NDCQ.

(c) $\tilde{L}(x, y, l_1, l_2) = x^2 + 2y^2 - l_1(x + 2y - 9) - l_2(-x^2 - y^2 + 16)$. (2 points).

(d) At the maximizer (x^*, y^*), there exist $l_1^* \ge 0, l_2^* \ge 0$ satisfy

$$\begin{cases} \frac{\partial \tilde{L}}{\partial x} &= 2x^* - l_1^* + 2x^* l_2^* \le 0 ...(1) \\ \frac{\partial \tilde{L}}{\partial y} &= 4y^* - 2l_1^* + 2y^* l_2^* \le 0 ...(2) \\ x^* \frac{\partial \tilde{L}}{\partial x} &= 0 = x^* \cdot (2x^* - l_1^* + 2x^* l_2^*) ...(3) \\ y^* \frac{\partial \tilde{L}}{\partial y} &= 0 = y^* \cdot (4y^* - 2l_1^* + 2y^* l_2^*) ...(4) \\ \frac{\partial \tilde{L}}{\partial \lambda_1} &= -(x^* + 2y^* - 9) \ge 0 ...(5) \\ \frac{\partial \tilde{L}}{\partial \lambda_2} &= (x^*)^2 + (y^*)^2 - 16 \ge 0 ...(6) \\ l_1^* \frac{\partial \tilde{L}}{\partial \lambda_1} &= 0 = l_1^* \cdot (x^* + 2y^* - 9) = 0 ...(7) \\ l_2^* \frac{\partial \tilde{L}}{\partial \lambda_2} &= 0 = l_2^* \cdot [(x^*)^2 + (y^*)^2 - 16] = 0 ...(8) \end{cases}$$

If you write down one correct equation, you get 0.5 point. If you write down all correct equations, you get 3 points.

(e) If $(x^*)^2 + (y^*)^2 = 16$, then $x^* + 2y^* \ne 9$.
(7) $\Rightarrow l_1^* = 0$.
(2) $\Rightarrow 2y^*(2 + l_2^*) \le 0$. Since $l_2^* \ge 0$ and $y^* \ge 0$, $y^* = 0$. It implies that $x^* = 4$.
(3) $\Rightarrow l_2^* = -1$. It is a contradiction. Therefore $(x^*)^2 + (y^*)^2 \ge 16$ can not be binding. If you get one of correct x^*, y^*, l_1^*, l_2^*, you get one point.

(f) If $x^* + 2y^* = 9$, then $(x^*)^2 + (y^*)^2 > 16$.
(8) $\Rightarrow l_2^* = 0$.(1 point)

$$(3), (4) \Rightarrow \begin{cases} x^*(2x^* - l_1^*) = 0 \\ y^*(4y^* - 2l_1^*) = 0. \end{cases}$$

If $x^* = 0$, then $y^* = \frac{9}{2} \Rightarrow l_1^* = 9$. It implies that $(x^*, y^*, l_1^*, l_2^*) = (0, \frac{9}{2}, 9, 0)$.(1 point)
If $y^* = 0$, then $x^* = 9 \Rightarrow l_1^* = 18$. It implies that $((x^*, y^*, l_1^*, l_2^*) = (9, 0, 18, 0)$.(1 point)
If $x^* > 0, y^* > 0$, then $l_1 = 2x^* = 2y^* \Rightarrow x^* = y^*$. Thus $3x^* = 9 \Rightarrow x^* = 3 \Rightarrow l_1^* = 6$. Thus $((x^*, y^*, l_1^*, l_2^*) = (3, 3, 6, 0)$.(2 points)

(g) $f(0, 9/2) = \frac{81}{2}$, $f(9, 0) = 81$, $f(3,3) = 27$. The maximum value is 81. (2 points)

4. (22%) Consider a C^2 utility function $U(x_1,x_2)$ such that $\frac{\partial U}{\partial x_1}>0$ and $\frac{\partial U}{\partial x_2}>0$. We want to maximize $U(x_1,x_2)$ under constraints $P_1x_1+P_2x_2\le I$, $x_1\ge 0$, and $x_2\ge 0$, where $P_1>0$, $P_2>0$ are unit prices and $I>0$ is the budget.

(a) (2%) Write down the usual version Lagrangian function.

(b) (2%) Show that the usual version NDCQ is satisfied.

(c) (3%) List the usual version first order conditions.

(d) (3%) From the first order conditions show that at the maximizer the constraint $P_1x_1+P_2x_2\le I$ must be binding.

(e) (6%) Suppose that the first order conditions has a solution (x_1^*,x_2^*) with $x_1^*>0$, $x_2^*>0$. Write down the bordered Hessian matrix at (x_1^*,x_2^*). List the condition which guarantees that $U(x_1^*,x_2^*)$ is a local maximum value.

(f) (6%) Now P_1, P_2 and I are parameters. Then the maximum value of U depends on P_1, P_2, and I, which is denoted by $\tilde{U}(P_1,P_2,I)$. Compute $\frac{\partial \tilde{U}}{\partial P_1}$, $\frac{\partial \tilde{U}}{\partial P_2}$, and $\frac{\partial \tilde{U}}{\partial I}$. Determine the signs of these partial derivatives.

Solution:

(a) $L(x_1,x_2,\lambda_1,\lambda_2,\lambda_3)=U(x_1,x_2)-\lambda_1(P_1x_1+P_2x_2-I)+\lambda_2x_1+\lambda_3x_2$.
(缺少 $\lambda_2x_1+\lambda_3x_2$ 扣一分。$\lambda_2x_1+\lambda_3x_2$ 寫成 $-\lambda_2x_1-\lambda_3x_2$ 扣一分)

(b) Let $g_1(x_1,x_2)=P_1x_1+P_2x_2$, $g_2(x_1,x_2)=-x_1$, $g_3(x_1,x_2)=-x_2$.
$\vec{\nabla} g_1=(P_1,P_2)\ne\vec{0}$, $\vec{\nabla} g_2=(-1,0)\ne\vec{0}$, $\vec{\nabla} g_3=(0,-1)\ne\vec{0}$. (1pt: 算出 $\vec{\nabla} g_1$, $\vec{\nabla} g_2$, $\vec{\nabla} g_3$)
At the maximizer at most two of the constraints are binding and any two of $\{\vec{\nabla} g_1,\vec{\nabla} g_2,\vec{\nabla} g_3\}$ are linearly independent. Hence the NDCQ is satisfied. (1pt)

(c) First Order conditions : At the maximizer (x_1^*,x_2^*) there are λ_1^*, λ_2^*, λ_3^* s.t

$$\begin{cases}\frac{\partial L}{\partial x_1}=\frac{\partial U}{\partial x_1}-\lambda_1^*P_1+\lambda_2^*=0\cdots ① \\ \frac{\partial L}{\partial x_2}=\frac{\partial U}{\partial x_2}-\lambda_1^*P_2+\lambda_3^*=0\cdots ② \\ \lambda_1^*(P_1x_1^*+P_2x_2^*-I)=0\cdots ③ \\ \lambda_2^*x_1^*=0\cdots ④ \\ \lambda_3^*x_2^*=0\cdots ⑤ \\ P_1x_1^*+P_2x_2^*\le I\cdots ⑥ \\ x_1^*\ge 0, x_2^*\ge 0, \lambda_1^*\ge 0, \lambda_2^*\ge 0, \lambda_3^*\ge 0\cdots ⑦\end{cases}$$

1pt: ①+②
1pt: ③+④+⑤
1pt: ⑥+⑦

(d) If $P_1x_1^*+P_2x_2^*<I$, then ③ $\Rightarrow \lambda_1^*=0$. (1pt) Thus ① $\Rightarrow \frac{\partial U}{\partial x_i}(x_1^*,x_2^*)+\lambda_2^*=0\Rightarrow \frac{\partial U}{\partial x_i}(x_1^*,x_2^*)=-\lambda_2^*\le 0$ contradiciton! (2pts)
Hence $P_1x_1^*+P_2x_2^*=I$.

(e) Since only the constraint $P_1x_1+P_2x_2\le I$ is binding at (x_1^*,x_2^*), the bordered Hessian matrix is $H=\begin{pmatrix}0 & P_1 & P_2\\ P_1 & L_{x_1x_1} & L_{x_1x_2}\\ P_2 & L_{x_2x_1} & L_{x_2x_2}\end{pmatrix}$

(1pt)

Because $L_{x_1x_1}=U_{x_1x_1}$, $L_{x_1x_2}=U_{x_1x_2}$, $L_{x_2x_2}=U_{x_2x_2}$, the bordered Hessian matrix is $\begin{pmatrix}0 & P_1 & P_2\\ P_1 & U_{x_1x_1} & U_{x_1x_2}\\ P_2 & U_{x_2x_1} & U_{x_2x_2}\end{pmatrix}$ (1pt)

There are 2 variables (x_1,x_2) and one binding constraint.

We need to check the last $(2-1)$ leading principal minor which is $\det\begin{pmatrix}0 & P_1 & P_2\\ P_1 & U_{x_1x_1} & U_{x_1x_2}\\ P_2 & U_{x_2x_1} & U_{x_2x_2}\end{pmatrix}$. (1pt)

$\det H=-P_1(P_1U_{x_2x_2}-P_2U_{x_2x_1})+P_2(P_1U_{x_2x_1}-P_2U_{x_1x_1})$
$=-P_1^2U_{x_2x_2}+2P_1P_2U_{x_1x_2}-P_2^2U_{x_1x_1}\Big|_{(x_1^*,x_2^*)}$. (2pts)

If $\det\begin{pmatrix}0 & P_1 & P_2\\ P_1 & U_{x_1x_1} & U_{x_1x_2}\\ P_2 & U_{x_2x_1} & U_{x_2x_2}\end{pmatrix}\Bigg|_{(x_1^*,x_2^*)}=-P_1^2U_{x_2x_2}+2P_1P_2U_{x_1x_2}-P_2^2U_{x_1x_1}>0$, then $U(x_1^*,x_2^*)$ is a local maximum value. (1pt)

(f) By the envelope theorem

$$\begin{aligned}\frac{\partial \tilde{U}}{\partial P_1} &= \frac{\partial L}{\partial P_1} = -\lambda_1^* x_1^* < 0 \\ \frac{\partial \tilde{U}}{\partial P_2} &= \frac{\partial L}{\partial P_2} = -\lambda_1^* x_2^* < 0 \\ \frac{\partial \tilde{U}}{\partial I} &= \lambda_1^* > 0\end{aligned}$$

(1pt for $\frac{\partial \tilde{U}}{\partial P_1} = -\lambda_1^* x_1^*$, 1pt for the sign$< 0$, 1pt for $\frac{\partial \tilde{U}}{\partial P_2} = -\lambda_1^* x_2^*$, 1pt for the sign$< 0$, 1pt for $\frac{\partial \tilde{U}}{\partial I} = \lambda_1^*$, 1pt for the sign$> 0$.)

⁎ (f) 的答案. $\frac{\partial \tilde{U}}{\partial P_1}$ 寫成 $-\lambda_1 x_1$ (沒有加*)，$\frac{\partial \tilde{U}}{\partial P_2} = -\lambda_1 x_2$(沒有加*)，$\frac{\partial \tilde{U}}{\partial I} = \lambda_1$(沒有加*) 不扣分

5. (18%) Suppose that in the following week you have 12 hours each day to study for the final exams of Calculus 4 and English. Let C be the number of hours per day spent studying for Calculsu 4 and E be the number of hours per day spent studying for English. Let your grade point average from these two courses be $GPA = f(C,E) = \frac{2}{3}\left(\sqrt{C}+\sqrt{2E}\right)$.

(a) (6%) Solve the optimization problem : Maximize $f(C,E)$ under the constraint $C+E=12$.
Use the bordered Hessian matrix to verify that the solution is indeed a local maximum.

(b) (6%) To assure that you obtain certain grades for Calculus 4 and English individually, you impose inequality constraints $C \geq 5$ and $E \geq 4$.
Solve the optimization problem : Maximize $f(C,E)$ under constraints $C+E=12$, $C\geq 5$, and $E \geq 4$.

(c) (6%) By the meaning of multipliers, estimate the maximum value of GPA when the constraints are $C+E=12.5$, $C \geq 5.5$ and $E \geq 4.1$.

Solution:

(a) Define $L(x,y,\mu) = \frac{2}{3}\left(\sqrt{C}+\sqrt{2E}\right) - \mu(C+E-12)$

$$\begin{cases} \frac{\partial L}{\partial C} = \frac{1}{3}\frac{1}{\sqrt{C}} - \mu = 0 \\ \frac{\partial L}{\partial E} = \frac{\sqrt{2}}{3}\frac{1}{\sqrt{E}} - \mu = 0 \\ \frac{\partial L}{\partial \mu} = C+E-12 = 0 \end{cases} \Rightarrow \begin{matrix} C = \frac{1}{9\mu^2} \\ E = \frac{2}{9\mu^2} \end{matrix}$$

and $C+E=12 \Rightarrow \mu = \frac{1}{6}$, $C=4$, $E=8$.

(4pts for correct answers $C=4$, $E=8$, $\mu=\frac{1}{6}$) The bordered Hessian matrix is $\hat{H} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & \frac{\partial^2}{\partial C^2}L & \frac{\partial^2}{\partial E \partial C}L \\ 1 & \frac{\partial^2}{\partial C \partial E}L & \frac{\partial^2}{\partial E^2}L \end{pmatrix}$ (1pt)

At $(C,E)=(4,8)$

$$\hat{H} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & -\frac{1}{48} & 0 \\ 1 & 0 & -\frac{1}{96} \end{pmatrix}.$$

$\det \hat{H} = \frac{1}{96} + \frac{1}{48} > 0$. (1pt) Hence $f(C,E)$ is a local maximum.

(b) Define $L(C,E,\mu,\lambda_1,\lambda_2) = \frac{2}{3}\left(\sqrt{C}+\sqrt{2E}\right) - \mu(C+E-12) - \lambda_1(-C+5) - \lambda_2(-E+4)$ (1pt) constraints are

$$\begin{aligned} h(C,E) &= C+E=12 \\ g_1(C,E) &= -C \leq -5 \\ g_2(C,E) &= -E \leq -4 \end{aligned}$$

$\vec{\nabla} h = (1,1)$, $\vec{\nabla} g_1 = (-1,0)$, $\vec{\nabla} g_2 = (0,-1)$. NDCQ is satisfied.

$$(2pts)\begin{cases} \frac{\partial L}{\partial C} = \frac{1}{3\sqrt{C}} - \mu + \lambda_1 = 0 \cdots ① \\ \frac{\partial L}{\partial E} = \frac{\sqrt{2}}{3\sqrt{E}} - \mu + \lambda_2 = 0 \cdots ② \\ \lambda_1(-C+5) = 0 \cdots ③ \\ \lambda_2(-E+4) = 0 \cdots ④ \\ C+E = 12 \cdots ⑤ \\ \lambda_1 \geq 0, \lambda_2 \geq 0, C \geq 5, E \geq 4. \end{cases}$$

We disscuss cases $\{\lambda_1>0,\lambda_2>0\}$, $\{\lambda_1=0,\lambda_2>0\}$, $\{\lambda_1>0,\lambda_2=0\}$, $\{\lambda_1=\lambda_2=0\}$
(以下4個 cases 共3分: 求出正確解，但是沒有討論全部的 cases 酌扣 1 2分)

1. $\{\lambda_1>0,\lambda_2>0\}$: ③④⑤ contradiciton.
2. $\{\lambda_1=0,\lambda_2>0\}$: ④⑤ $\Rightarrow E=4$, $C=8$
   ① $\Rightarrow \mu = \frac{1}{6\sqrt{2}}$
   ② $\Rightarrow \frac{\sqrt{2}}{6} - \frac{1}{6\sqrt{2}} + \lambda_2 = 0 \Rightarrow \lambda_2 < 0$ contradiction.
3. $\{\lambda_1>0,\lambda_2=0\}$: ③⑤ $\Rightarrow C=5$, $E=7$, ② $\Rightarrow \mu = \frac{1}{3}\sqrt{\frac{2}{7}}$
   ① $\Rightarrow \lambda_1 = \mu - \frac{1}{3\sqrt{5}} = \frac{1}{3}\sqrt{\frac{2}{7}} - \frac{1}{3\sqrt{5}} > 0$.
   Ans : $(C^*,E^*,\mu^*,\lambda_1^*,\lambda_2^*) = \left(5,7,\frac{1}{3}\sqrt{\frac{2}{7}},\frac{1}{3}\sqrt{\frac{2}{7}}-\frac{1}{3\sqrt{5}},0\right)$.

4. $\{\lambda_1 = 0, \lambda_2 = 0\}$: ①②⑤ $\Rightarrow C = 4,\ E = 8$ contradiction to the constraint $C \geq 5$.

Hence the maximum value is $f(5,7) = \frac{2}{3}(\sqrt{5} + \sqrt{14})$.

(c) The maximum value for new constraints can be approximated by

$$f(5,7) + \mu^* \times (12.5 - 12) + (-\lambda_1^*) \times (5.5 - 5) + (-\lambda_2^*) \times (4.1 - 4) \quad (4pts)$$
$$= f(5,7) + 0.5(\mu^* - \lambda_1^*) - 0.1\lambda_2^* = \frac{2}{3}(\sqrt{5} + \sqrt{14}) + \frac{1}{6\sqrt{5}} \quad (2pts)$$

6. (12%) Suppose that $f(x,y,z)$ is a C^2 function and $P=(1,2,-3)$ is a critical point of f which means that $f_x=f_y=f_z=0$ at P. If at point P, $f_{xx}=1$, $f_{yy}=1$, $f_{zz}=-3$, $f_{xy}=-1$, $f_{xz}=3$, and $f_{yz}=3$, determine whether $f(1,2,-3)$ is a local extreme value.

(a) (1%) Write down the Hessian matrix of f at P which is denoted by H.

(b) (3%) Is H positive definite, negative definite, or indefinite? Is $f(1,2,-3)$ a local maximum, local minimum, or a saddle point?

(c) (3%) Now we want to find the extreme value of f under the constraint $x^2+xy+xz-3z=9$. Write down the Lagrangian function $\mathcal{L}(x,y,z,\mu)$. Find μ^* such that $\frac{\partial}{\partial x}\mathcal{L}=\frac{\partial}{\partial y}\mathcal{L}=\frac{\partial}{\partial z}\mathcal{L}=0$ when $(x,y,z,\mu)=(1,2,-3,\mu^*)$.

(d) (5%) On the constraint set $x^2+xy+xz-3z=9$, is $f(1,2,-3)$ a local maximum, local minimum, or a saddle point?

Solution:

(a) $H=\begin{pmatrix}1 & -1 & 3\\ -1 & 1 & 3\\ 3 & 3 & -3\end{pmatrix}$ (1pt)

(b) The leading principal minors of H are $H_1=1$, $H_2=\begin{vmatrix}1 & -1\\ -1 & 1\end{vmatrix}=0$, $H_3=\begin{vmatrix}1 & -1 & 3\\ -1 & 1 & 3\\ 3 & 3 & -3\end{vmatrix}=-36$ (2pts)

$\because H_1\cdot H_3=-36<0 \therefore H$ is indefinite. $f(P)$ is a saddle point. (1pt)

(c) $\mathcal{L}(x,y,z,\mu)=f(x,y,z)-\mu(x^2+xy+xz-3z-9)$ (1pt)
At $(1,2,-3)$,

$$\begin{aligned}\frac{\partial L}{\partial x}&=\frac{\partial f}{\partial x}(1,2,-3)-\mu\times 1=-\mu\\ \frac{\partial L}{\partial y}&=\frac{\partial f}{\partial y}(1,2,-3)-\mu\times 1=-\mu\\ \frac{\partial L}{\partial z}&=\frac{\partial f}{\partial z}(1,2,-3)-\mu\times(-2)=2\mu\end{aligned}$$

when $\mu=0$, $\frac{\partial L}{\partial x}=\frac{\partial L}{\partial y}=\frac{\partial L}{\partial z}=0$. Hence $\mu^*=0$. (2pts)

(d) $S: g(x,y,z)=9$ where $g(x,y,z)=x^2+xy+xz-3z$, $\vec{\nabla}g(1,2,-3)=(1,1,-2)$.
At $(x,y,z,\mu)=(1,2,-3,0)$, $L_{xx}=f_{xx}=1$, $L_{xy}=f_{xy}=-1$, $L_{xz}=f_{xz}=3$, $L_{yy}=f_{yy}=1$, $L_{yz}=f_{yz}=3$, $L_{zz}=f_{zz}=-3$. (1pt) The bordered Hessian is

$$\hat{H}=\begin{pmatrix}0 & 1 & 1 & -2\\ 1 & 1 & -1 & 3\\ 1 & -1 & 1 & 3\\ -2 & 3 & 3 & -3\end{pmatrix}\quad (1pt)$$

There are 3 variables and 1 constraint. Hence we need to check the last $3-1$ leading principal minors of $\hat{H}$.

$$\hat{H}_3=\begin{vmatrix}0 & 1 & 1\\ 1 & 1 & -1\\ 1 & -1 & 1\end{vmatrix}=-4<0\quad (1pt)$$

$$\hat{H}_4=\begin{vmatrix}0 & 1 & 1 & -2\\ 1 & 1 & -1 & 3\\ 1 & -1 & 1 & 3\\ -2 & 3 & 3 & -3\end{vmatrix}=-36<0\quad (1pt)$$

$\hat{H}_3$ and $\hat{H}_4$ both have the same sign as $(-1)^1$. Thus H is positive definite on the constraint set. $f(1,2,-3)$ is a local minimum on S. (1pt)