微乙小考五 (2020/06/11)

1. (4 pts) 用泰勒展式法解微分方程 $y'(t) - ty(t) = 0$ 的一般解。 (其他方法一律不計分)

sol: 假設$y(t)$對$t=0$有泰勒展式

$$y(t) = a_0 + a_1 t + a_2 t^2 + \cdots + a_n t^n + \dots$$

則

$$\begin{array}{rcccccccccccc} y'(t) & = & a_1 & + & 2a_2 t & + & 3a_3 t^2 & + & \dots & + & (n+1)a_{n+1}t^n & + & \dots \\ ty(t) & = & & & a_0 t & + & a_1 t^2 & + & \dots & + & a_{n-1}t^n & + & \dots \end{array}$$

$y'(t) = ty(t)$，比較兩邊係數得

$$\begin{array}{rcccc} a_1 & = & 0 & & \\ a_2 & = & \frac{1}{2}a_0 & & \\ a_3 & = & \frac{1}{3}a_1 & = & 0 \\ a_4 & = & \frac{1}{4}a_2 & = & \frac{1}{2\cdot 4}a_0 \\ \vdots & = & \vdots & = & \vdots \\ a_{2k-1} & = & \frac{1}{2k-1}a_{2k-3} & = & 0 \\ a_{2k} & = & \frac{1}{2k}a_{2k-2} & = & \frac{1}{2^k k!}a_0 \\ \vdots & = & \vdots & = & \vdots \end{array}$$

所以

$$\begin{aligned} y(t) &= a_0\left(1 + \frac{1}{2}t^2 + \frac{1}{2\cdot 4}t^4 + \cdots + \frac{1}{2^k k!}t^{2k} + \cdots\right) \\ &= a_0\left(e^{\frac{1}{2}t^2}\right) \end{aligned}$$

2. (4 pts) 令 $(t_0, y_0) = (0, \frac{1}{2})$, $\Delta t = \frac{1}{2}$, 用歐拉法解 $y'(t) = y(t)(y(t)-1)$, 求出 (t_1, y_1), (t_2, y_2). 答案以分數表示 (不用計算小數值).

sol:

$$y'(t) = f(t, y(t)) = y(t)(y(t)-1)$$

$$\begin{cases} t_1 = t_0 + \Delta t = \frac{1}{2} \\ y_1 = f(t_0, y_0)\Delta t + y_0 = \frac{3}{8} \end{cases}$$

$$\begin{cases} t_2 = t_1 + \Delta t = 1 \\ y_2 = f(t_1, y_1)\Delta t + y_1 = \frac{33}{128} \end{cases}$$

3. (6 pts) 計算 (1) $\int_{-2}^{1} \frac{1}{x^3}\, dx$ (2) $\int_0^\infty e^{-3x}\, dx$ (3) $\int_0^\infty x^{\frac{1}{2}} e^{-x^3}\, dx$.

sol: (1)

$$
\begin{aligned}
\int_{-2}^{1} \frac{1}{x^3}\,dx &= \int_{-2}^{0^-} \frac{1}{x^3}\,dx + \int_{0^+}^{1} \frac{1}{x^3}\,dx \\
&= \lim_{a\to 0^-} \int_{-2}^{a} \frac{1}{x^3}\,dx + \lim_{a\to 0^+} \int_{a}^{1} \frac{1}{x^3}\,dx \\
&= \lim_{a\to 0^-} \left.\frac{-1}{2x^2}\right|_{-2}^{a} + \lim_{a\to 0^+} \left.\frac{-1}{2x^2}\right|_{a}^{1} \\
&= \lim_{a\to 0^-} \left(\frac{-1}{2a^2} + \frac{1}{8}\right) + \lim_{a\to 0^+} \left(\frac{-1}{2} + \frac{1}{2a^2}\right) \\
&\quad \therefore \text{不存在}
\end{aligned}
$$

(2)

$$
\begin{aligned}
\int_{0}^{\infty} e^{-3x}\,dx &= \lim_{a\to\infty} \int_{0}^{a} e^{-3x}\,dx \\
&= \lim_{a\to\infty} \left.\frac{-e^{-3x}}{3}\right|_{0}^{a} \\
&= \lim_{a\to\infty} \left(\frac{-e^{-3a}}{3} + \frac{1}{3}\right) \\
&= \frac{1}{3}
\end{aligned}
$$

(3)

$$
\begin{aligned}
\int_{0}^{\infty} x^{\frac{1}{2}} e^{-x^3}\,dx &\overset{u=x^{\frac{3}{2}}}{=} \frac{2}{3}\int_{0}^{\infty} e^{-u^2}\,du \\
&= \frac{2}{3}\cdot\frac{\sqrt{\pi}}{2} \\
&= \frac{\sqrt{\pi}}{3}
\end{aligned}
$$

4. (6 pts) 計算 $\int_{-\infty}^{\infty} x^2 e^{-x^2}\,dx$ 和 $\int_{-\infty}^{\infty} x^2 e^{-\frac{(x-\alpha)^2}{\beta^2}}\,dx$, α, β 為常數.

sol:

$$
\begin{aligned}
\int_{-\infty}^{\infty} x^2 e^{-x^2}\,dx &= 2\int_{0}^{\infty} x^2 e^{-x^2}\,dx \\
&= 2\lim_{a\to\infty} \int_{0}^{a} x^2 e^{-x^2}\,dx \\
&= 2\lim_{a\to\infty} \int_{0}^{a} x\,d\left(\frac{-e^{-x^2}}{2}\right) \\
&= 2\lim_{a\to\infty} \left(\left.\frac{-xe^{-x^2}}{2}\right|_{0}^{a} + \frac{1}{2}\int_{0}^{a} e^{-x^2}\,dx\right) \\
&= 2\left(0 + \frac{\sqrt{\pi}}{4}\right) \\
&= \frac{\sqrt{\pi}}{2}
\end{aligned}
$$

$$
\begin{aligned}
\int_{-\infty}^{\infty} x^2 e^{-\frac{(x-\alpha)^2}{\beta^2}}\,dx &\overset{u=\frac{x-\alpha}{\beta}}{=} \int_{-\infty}^{\infty} \beta\,(\beta u+\alpha)^2 e^{-u^2}\,du \\
&= \beta^3 \int_{-\infty}^{\infty} u^2 e^{-u^2}\,du + 2\alpha\beta^2 \int_{-\infty}^{\infty} u e^{-u^2}\,du + \alpha^2\beta \int_{-\infty}^{\infty} e^{-u^2}\,du \\
&= \frac{\beta^3\sqrt{\pi}}{2} + 0 + \alpha^2\beta\sqrt{\pi} \\
&= \sqrt{\pi}\beta\left(\alpha^2 + \frac{\beta^2}{2}\right)
\end{aligned}
$$