微乙小考四 (2020/5/28)

1. (10 %) 解微分方程 $y'(t) = 1 + t^3 + y^2(t) + t^3y^2(t)$.

sol:
$$\frac{dy}{dt} = (1+t^3)(1+y^2) \implies \frac{dy}{(1+y^2)} = (1+t^3)dt \overset{\int}{\implies} \tan^{-1} y = t + \frac{t^4}{4} + C$$

$$\implies y = \tan\left(t + \frac{t^4}{4} + C\right), \text{for some } C \in \mathbb{R}.$$

2. (10 %) 解微分方程
$$\begin{cases} ty'(t) = 2y + t^3 \sec(t)\tan(t),\ 0 < t < \dfrac{\pi}{2} \\ y(\pi/3) = 2. \end{cases}$$

sol: *Step 1. Solve the equation*

We can rearrange the above equation to get $ty' - 2y = t^3 \sec(t)\tan(t)$.

Observing that $(t^{-2}y)' = t^{-2}y' - 2t^{-3}y$, we have $t^3(t^{-2}y)' = ty' - 2y = t^3 \sec(t)\tan(t)$.

Since $t \neq 0$, we can eliminate the t^3,

$$(t^{-2}y)' = \sec(t)\tan(t) \overset{\int}{\implies} t^{-2}y = \sec(t) + C \implies y = (\sec(t) + C)t^2, \text{for some } C \in \mathbb{R}.$$

(Or we can rewrite the above differential equation in the standard form $y'(t)+p(t)y(t) = q(t)$, says $y' - 2t^{-1}y = t^2 \sec(t)\tan(t)$, and then use the formula $y = e^{-\int pdt} \int e^{\int pdt} qdt$ to obtain the same result.

$$e^{\int pdt} = e^{\int -2t^{-1}dt} = C_1t^{-2}, C_1 \in \mathbb{R}$$

$$y = \frac{\int e^{\int pdt} qdt}{e^{\int pdt}} = \frac{\int C_1t^{-2}t^2 \sec(t)\tan(t)dt}{C_1t^{-2}} = (\sec(t) + C_2)t^2, C_2 \in \mathbb{R}.)$$

Step 2. Satisfy the condition

To satisfy the condition $y(\frac{\pi}{3}) = 2$, let’s plug $(t, y) = (\frac{\pi}{3}, 2)$ into the above equation

$$2 = \left(\sec\left(\frac{\pi}{3}\right) + C\right)\frac{\pi^2}{9} \implies C = \frac{18}{\pi^2} - 2.$$

Finally, the answer is $y = (\sec(t) + (\frac{18}{\pi^2} - 2))t^2$.