1. (14%) 求 $f(x,y)=\dfrac{x}{x^2+y^2+1}$ 的候選點及其極值性質.

Solution:

Find (x,y) such that

$$\begin{cases} f_x = \dfrac{x^2+y^2+1-2x^2}{(x^2+y^2+1)^2} = \dfrac{-x^2+y^2+1}{(x^2+y^2+1)^2} = 0 & -(1) \\ f_y = \dfrac{-2xy}{(x^2+y^2+1)^2} = 0, & -(2) \end{cases} \quad \text{(4 points)}$$

The equation (2) implies that $x=0$ or $y=0$. But if $x=0$, then $y^2+1\neq 0$. If $y=0$, then $x=\pm 1$. So critical points are $(1,0)$ and $(-1,0)$. (2 points)

Compute that

$$f_{xx} = \frac{-2x(x^2+y^2+1)^2-(-x^2+y^2+1)\cdot 2\cdot (x^2+y^2+1)\cdot 2x}{(x^2+y^2+1)^4}, \text{ (1 points)}$$

$$f_{yy} = \frac{-2x(x^2+y^2+1)^2+2xy\cdot 2\cdot (x^2+y^2+1)\cdot 2y}{(x^2+y^2+1)^4}, \text{ (1 points)}$$

$$f_{xy} = \frac{2y(x^2+y^2+1)^2-(-x^2+y^2+1)\cdot 2\cdot (x^2+y^2+1)\cdot (2y)}{(x^2+y^2+1)^4}. \text{ (1 points)}$$

This implies that $f_{xx}(1,0)=f_{yy}(1,0)=-\dfrac{1}{2}$, $f_{xx}(-1,0)=f_{yy}(-1,0)=\dfrac{1}{2}$ and $f_{xy}(\pm 1,0)=0$.

Let $D(x,y)=f_{xx}f_{yy}-f_{xy}^2$. (1 points)

Since $D(1,0)>0$ and $f_{xx}(1,0)<0$, $f(1,0)=\dfrac{1}{2}$ is a local maximum. (2 points)

Since $D(-1,0)>0$ and $f_{xx}(-1,0)>0$, $f(-1,0)=-\dfrac{1}{2}$ is a local minimum. (2 points)

2. (15%) $x^2+y^2=1$. 以 Lagrange 乘子法求 x^2y-y^2 的最大值與最小值。

Solution:

由Lagrange乘子法

$$\begin{cases} 2xy &= \lambda\cdot 2x \\ x^2-2y &= \lambda\cdot 2y \\ x^2+y^2 &= 1 \end{cases} \quad \text{5 pts for correct equations.}$$

由第一式得 $x=0$ 或 $y=\lambda$.

A. $x=0$. 由第三式，$y=\pm1$. 代入 $x^2y-y^2=0-1=-1$. (3 pts for solutions $(x,y)=(0,\pm1)$)

B. $y=\lambda$. (5 pts for complete discussions on this case.)
由第二式得，$x^2=2\lambda^2+2\lambda$. 再代入第三式得

$$2\lambda^2+2\lambda+\lambda^2=1 \Rightarrow 3\lambda^2+2\lambda-1=0 \Rightarrow \lambda=\frac{1}{3} \text{ 或 } -1.$$

a. $\lambda=\frac{1}{3}$. $y=\lambda=\frac{1}{3}$, 且 $x=\pm\frac{2\sqrt{2}}{3}$. 代入 $x^2y-y^2=\frac{8}{9}\cdot\frac{1}{3}-\frac{1}{9}=\frac{5}{27}$.

b. $\lambda=-1$. $y=\lambda=-1$ 且 $x=0$， 與前同。

故知在 $(\pm\frac{2\sqrt{2}}{3},\frac{1}{3})$ 得最大值 $\frac{5}{27}$; 在 $(0,\pm1)$ 得最小值 -1。 (2 pts for final answers.)

3. (14%) 令 $F(u,v) = \left(u, \dfrac{v}{u+1}\right)$ 且 Σ 為一個在 uv 座標平面裡的平行四邊形區域，此區域的四個頂點分別為

$$(0,0), \quad (2,0), \quad (1,1), \quad (3,1).$$

令 $\Omega = \{F(u,v) : (u,v) \in \Sigma\}$ 為 Σ 經由 F 映射所形成的區域。 求出 Ω 的面積 (可直接使用 $\int \ln x \, dx = x\ln x - x + c$)。

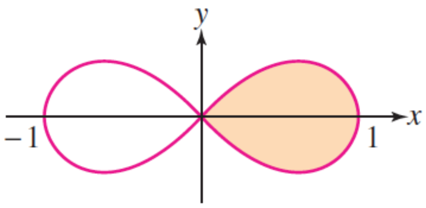
Solution:

Let $x = u$ and $y = \dfrac{v}{u+1}$. We have

$$J(u,v) = \begin{vmatrix} 1 & \dfrac{-v}{(u+1)^2} \\ 0 & \dfrac{1}{u+1} \end{vmatrix} = \frac{1}{u+1}. \quad (4\%)$$

$$\begin{aligned}
\text{Area}(\Omega) &= \iint_{\Omega} 1 \, dA \\
&= \iint_{\Sigma} \frac{1}{|u+1|} \, dA(u,v) \quad (2\%) \\
&= \int_0^1 \int_v^{v+2} \frac{1}{u+1} \, du \, dv \quad (4\%) \\
&= \int_0^1 \ln(u+1)\Big|_v^{v+2} \, dv \quad (2\%) \\
&= \int_0^1 (\ln(v+3) - \ln(v+1)) \, dv \\
&= ((v+3)\ln(v+3) - (v+3) - (v+1)\ln(v+1) + v + 1)\Big|_0^1 \\
&= 3\ln 4 - 3\ln 3. \quad (2\%).
\end{aligned}$$

4. (15%) 令 Ω 為 $r^2 = \cos 2\theta$ 在一、四象限所圍成的區域。求 $\iint_\Omega x^2 + y^2 \mathrm{d}A$.



Solution:

Solution 1:

$$\Omega = \{(r,\theta) \big| \underbrace{-\frac{\pi}{4} \le \theta \le \frac{\pi}{4}}_{3\text{ pts}}, \underbrace{0 \le r \le \sqrt{\cos 2\theta}}_{2\text{ pts}}\}$$

$$\iint_\Omega x^2 + y^2 \mathrm{d}A$$

$$= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \int_0^{\sqrt{\cos 2\theta}} r^2 \cdot r\mathrm{d}r\ \mathrm{d}\theta \quad \leftarrow \quad 4 \text{ pts}$$

(2 pts for the Jacobian= r, 2 pts for correct order $\mathrm{d}r\ \mathrm{d}\theta$ and upper and lower bounds.)

$$= \frac{1}{4} \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \cos^2 2\theta \mathrm{d}\theta \quad \leftarrow \quad 2 \text{ pts}$$

$$= \frac{1}{4} \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{1 + \cos 4\theta}{2} \mathrm{d}\theta \quad \leftarrow \quad 2 \text{ pts}$$

$$= \frac{1}{8} \left[\theta + \frac{1}{4} \sin 4\theta\right]\Bigg|_{\theta=-\frac{\pi}{4}}^{\theta=\frac{\pi}{4}} = \frac{\pi}{16} \quad \leftarrow \quad 2 \text{ pts}$$

Solution 2:

Let R be the part of Ω in the first quadrant.

By Symmetry, $\iint_\Omega x^2 + y^2 \mathrm{d}A = 2 \iint_R x^2 + y^2 \mathrm{d}A$

$R = \{(r,\theta) | 0 \le \theta \frac{\pi}{4}, 0 \le r \le \sqrt{\cos 2\theta}\}$ (5 pts)

$$\iint_\Omega x^2 + y^2 \mathrm{d}A = 2 \iint_R x^2 + y^2 \mathrm{d}A = 2 \int_0^{\frac{\pi}{4}} \int_0^{\sqrt{\cos 2\theta}} r^2 \cdot r\mathrm{d}r\ \mathrm{d}\theta \quad \leftarrow \quad 4 \text{ pts}$$

$$= \frac{1}{4} \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \cos^2 2\theta \mathrm{d}\theta \quad \leftarrow \quad 2 \text{ pts}$$

$$= \frac{1}{4} \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{1 + \cos 4\theta}{2} \mathrm{d}\theta \quad \leftarrow \quad 2 \text{ pts}$$

$$= \frac{1}{8} \left[\theta + \frac{1}{4} \sin 4\theta\right]\Bigg|_{\theta=-\frac{\pi}{4}}^{\theta=\frac{\pi}{4}} = \frac{\pi}{16} \quad \leftarrow \quad 2 \text{ pts}$$