1. (17%) 假設一物體溫度的變化率與「該溫度及室溫溫差」成正比且正負號相反。令 $T(t)$ 為此物在時間 t 的溫度，$T(t_0) = T_0$，室溫是 H。
   (a) (3%) 寫下 $T(t)$ 滿足的微分方程式。
   (b) (8%) 求 $T(t)$ 的解。
   (c) (6%) 將肉從 $-15°C$ 的冷凍庫拿出來在室溫 $25°C$ 下解凍，經過一小時其溫度為 $5°C$。相同的解凍過程在室溫 $10°C$ 的冬天需費時多久？

Solution:

(a) $T'(t) = -\alpha(T(t) - H)$ for some $\alpha > 0$.

(b) Suppose $T(t) \neq H$. $\dfrac{T'(t)}{T(t) - H} = -\alpha \Rightarrow \ln|T(t) - H| = -\alpha t + C$.
$|T(t) - H| = Ke^{-\alpha t}$, where K is a constant.
$T(t_0) = T_0 \Rightarrow |T(t) - H| = |T_0 - H|e^{-\alpha(t-t_0)}$.

① If $T_0 > H$, then $T(t) > H$ for all t and $T(t) = H + (T_0 - H)e^{-\alpha(t-t_0)}$.
② If $T_0 < H$, then $T(t) < H$ for all t and $T(t) = H + (T_0 - H)e^{-\alpha(t-t_0)}$.
③ If $T_0 = H$, then $T(t) = H$

(c) Given that $T_0 = T(0) = -15$, $H = 25$, we know that $T(t) = 25 + (-15 - 25)e^{-\alpha t} = 25 - 40e^{-\alpha t}$
$T(1) = 5 \Rightarrow 5 = 25 - 40e^{-\alpha \cdot 1} \Rightarrow e^{-\alpha} = \dfrac{1}{2}$.
Now $H = 10$, $T(t) = 10 + (-15 - 10)e^{-\alpha t} = 10 - 25\left(\dfrac{1}{2}\right)^t$
$T(t) = 5 \Rightarrow \dfrac{1}{5} = \left(\dfrac{1}{2}\right)^t \Rightarrow t = \log_2 5$
Ans: 需 $\log_2 5$ 小時。

2. (12%) 解微分方程

$$\begin{cases} y'(t) = \dfrac{e^{-2y}t^3 - e^{-2y}}{t^2}, & t > 0 \\ y(1) = 0. \end{cases}$$

Solution:

$y'(t) = \dfrac{t^3 - 1}{t^2} e^{-2y} \Rightarrow \dfrac{y'(t)}{e^{-2y}} = \dfrac{t^3 - 1}{t^2}$ is separable ODE.

$$\begin{aligned}
&\Rightarrow \quad \int e^{2y} dy = \int t - \frac{1}{t^2} dt \\
&\Rightarrow \quad \frac{1}{2} e^{2y} = \frac{1}{2} t^2 + \frac{1}{t} + C \text{ since } y(1) = 0 \text{ we have } C = -1 \\
&\Rightarrow \quad 2y = \ln(t^2 + 2t^{-1} - 2) \\
&\Rightarrow \quad y = \frac{1}{2} \ln(t^2 + 2t^{-1} - 2)
\end{aligned}$$

3. (12%) 解微分方程

$$\begin{cases} (t^2+1)y'(t) - (t^3+t)y(t) = e^{\frac{t^2}{2}}, \\ y(0) = 1. \end{cases}$$

Solution:

$$\begin{aligned}
& (t^2+1)y'(t) - (t^3+t)y(t) = e^{\frac{t^2}{2}} \\
\Rightarrow\quad & y'(t) - ty(t) = \frac{e^{\frac{t^2}{2}}}{t^2+1} \\
\Rightarrow\quad & e^{-\frac{t^2}{2}}y'(t) - te^{-\frac{t^2}{2}}y(t) = \frac{1}{t^2+1} \\
\Rightarrow\quad & (e^{-\frac{t^2}{2}}y(t))' = \frac{1}{t^2+1} \\
\Rightarrow\quad & e^{-\frac{t^2}{2}}y(t) = \int \frac{1}{t^2+1}dt = \tan^{-1} t + C \text{ since } y(0) = 1 \text{ we have } C = 1 \\
\Rightarrow\quad & y(t) = e^{\frac{t^2}{2}}(\tan^{-1} t + 1)
\end{aligned}$$

4. (16%) 令 $y(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3 + \cdots$ 是微分方程

$$\begin{cases} \dfrac{dy}{dt} = (t+1)y, \\ y(0) = 1. \end{cases}$$

以泰勒方法所得之解。

(a) (6%) 求出 a_0, a_1, a_2, a_3 的值。

(b) (4%) 使用此泰勒多項式 $a_0 + a_1 t + a_2 t^2 + a_3 t^3$ 求出 $y(0.2)$ 的近似值，四捨五入到小數點下第三位。

(c) (6%) 令 $(t_0, y_0) = (0, 1)$ 和 $h = \Delta t = 0.1$. 使用歐拉法求出 y_1 和 y_2.

Solution:

(a)

$$\begin{cases} \dfrac{dy}{dt} = a_1 + 2a_2 t + 3a_3 t^2 + \cdots \\ (t+1)y = a_0(t+1) + a_1 t(t+1) + a_2 t^2 (t+1) + \cdots \end{cases} \Rightarrow \begin{cases} \dfrac{dy}{dt} = a_1 + 2a_2 t + 3a_3 t^2 + \cdots \\ (t+1)y = a_0 + (a_0 + a_1)t + (a_1 + a_2)t^2 + \cdots \end{cases}$$

i.e the coefficient of t^n is $(n+1)a_{n+1} = a_{n-1} + a_n$ for all $n \geq 1$ and $a_1 = a_0 = y(0) = 1$, $\Rightarrow a_2 = 1$, $a_3 = \dfrac{2}{3}$

(b) $y(0.2) = 1 + 1 \cdot 0.2 + 1 \cdot 0.2^2 + \dfrac{2}{3} \cdot 0.2^3 = 1 + 0.2 + 0.04 + 0.0053 = 1.2453$

(c) By Eular Method $y_{n+1} - y_n = y'(t_n)(t_{n+1} - t_n)$

$$\begin{aligned} & y_1 - y_0 = y'(t_0)(t_1 - t_0) \\ \Rightarrow \quad & y_1 = y_0 + y'(t_0)\Delta t \\ \Rightarrow \quad & y_1 = 1 + 1 \cdot 0.1 = 1.1 \end{aligned}$$

and

$$\begin{aligned} & y_2 - y_1 = y'(t_1)(t_2 - t_1) \\ \Rightarrow \quad & y_2 = y_1 + y'(t_1)\Delta t \\ \Rightarrow \quad & y_2 = 1.1 + (0.1 + 1)1.1 \cdot 0.1 = 1.1 + 0.121 = 1.221 \end{aligned}$$

5. (14%) 已知 $\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$，計算以下的瑕積分。

(a) (7%) $\int_0^{\infty} \sqrt{x} e^{-x} dx.$

(b) (7%) $\int_{-\infty}^{\infty} e^{2x} e^{-x^2} dx$ （提示：$-x^2 + 2x = -(x-1)^2 + 1$。）

Solution:

(a)

$$\begin{aligned}\int_0^{\infty} \sqrt{x} e^{-x} dx &= \lim_{b\to\infty} \int_1^b \sqrt{x} e^{-x} dx \underset{\left(\begin{array}{l} u = \sqrt{x}, \\ du = \frac{1}{2\sqrt{x}} dx \end{array}\right)}{=} \lim_{b\to\infty} \int_0^{\sqrt{b}} 2u^2 e^{-u^2} du \\ &= \lim_{b\to\infty} u(-e^{-u^2})\Big|_{u=0}^{u=\sqrt{b}} + \int_0^{\sqrt{b}} e^{-u^2} du \\ &= \lim_{b\to\infty} \int_0^{\sqrt{b}} e^{-u^2} du = \int_0^{\infty} e^{-u^2} du = \frac{\sqrt{\pi}}{2}\end{aligned}$$

(b) $\int_{-\infty}^{\infty} e^{2x} e^{-x^2} dx = \int_{-\infty}^{\infty} e^{-(x-1)^2+1} dx = e \int_{-\infty}^{\infty} e^{-(x-1)^2} dx$

By change of variable $u = x - 1$, $\int_{-\infty}^{\infty} e^{-(x-1)^2} dx = \int_{-\infty}^{\infty} e^{-u^2} dx = \sqrt{\pi}$

Hence, $\int_{-\infty}^{\infty} e^{2x} e^{-x^2} dx = e\sqrt{\pi}$.

6. (17%) $f_X(t) = C\,t^7 e^{-t},\ t \geq 0$ 是隨機變數X的機率密度函數。

(a) (7%) 求 C.

(b) (10%) 求其期望值 $\mathbf{E}(X)$ 和變異數 $\mathbf{Var}(X)$.

Solution:

(a)

$$\int_0^\infty C\,t^7 e^{-t} dt = C\int_0^\infty t^{8-1}e^{-t}dt = C\Gamma(8) = C\cdot 7! = 1$$

所以 $C = \frac{1}{7!}$, 且 $f_X(t) = \frac{1}{7!}\,t^7 e^{-t}$

(b) $\mathbf{E}(X)$ 和變異數 $\mathbf{Var}(X)$.

$$\mathbf{E}(X) = \int_0^\infty t\cdot\frac{1}{7!}\,t^7 e^{-t}\,dt = \frac{1}{7!}\int_0^\infty t^8 e^{-t}\,dt = \frac{8!}{7!} = 8$$

$$\mathbf{E}(X^2) = \int_0^\infty t^2\cdot\frac{1}{7!}\,t^7 e^{-t}\,dt = \frac{1}{7!}\int_0^\infty t^9 e^{-t}\,dt = \frac{9!}{7!} = 72$$

所以

$$\mathbf{Var}(X) = \mathbf{E}(X^2) - \mathbf{E}(X)^2 = 72 - 64 = 8$$

評分建議. 分數的分配建議為1 : 1 : 1. 能清楚寫出定義，可給基本分。知道gamma積分定義，給基本分。由於是gamma函數，可不提瑕積分收斂性。

7. (12%) $f_X(t) = \frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{(t-\mu)^2}{2\sigma^2}}$, $t \in \mathbb{R}$ 是隨機變數 X 的機率密度函數，其中 μ, σ 是常數，$\sigma > 0$. 令隨機變數 $Y = \frac{X-\mu}{\sigma}$. 依照下述方法求出Y的機率密度函數 $f_Y(s)$

(a) (3%) 說明 $\mathbf{P}(Y \le s) = \mathbf{P}(X \le \mu + \sigma s)$

(b) (3%) 上述所對應的積分等式為 $\int_{-\infty}^{s} f_Y(y)\,dy =$ ________________.

(c) (6%) 假設可以使用微積分基本定理，求出 $f_Y(s)$.

Solution:

(a) 因為

$$\mathbf{P}(Y \le s) = \mathbf{P}(\frac{X-\mu}{\sigma} \le s) = \mathbf{P}(X \le \mu + \sigma s)$$

(b) $\int_{-\infty}^{s} f_Y(y)\,dy =$

$$\int_{-\infty}^{\mu+\sigma s} \frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{(t-\mu)^2}{2\sigma^2}}\,dt \qquad \text{或} \qquad \int_{-\infty}^{\mu+\sigma s} f_X(t)dt$$

(c)

$$f_Y(s) = \left(\int_{-\infty}^{s} f_Y(y)\,dy\right)' = \left(\int_{-\infty}^{\mu+\sigma s} \frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{(t-\mu)^2}{2\sigma^2}}\,dt\right)' = \frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{s^2}{2}} \cdot \sigma = \frac{1}{\sqrt{2\pi}}e^{-\frac{s^2}{2}}$$

評分建議. 分數的分配建議為1 : 1 : 3. 會用微積分基本定理有一定分數，計算正確再給剩下分數。