1082 微乙01-04班期末考詳解及評分標準

1. (17%) 一個八卦消息在 400人的公司中傳開，在 $9:00$ 有 10 人知道消息，在 $12:00$ 有 100 人聽到耳語。令 $y(t)$ 為 $9:00$ 後第 t 小時公司裡聽過八卦消息的人數比例。假設八卦傳播的速率與 $y(1-y)$ 成正比。
   (a) (10%) 寫下 $y(t)$ 滿足的微分方程式並求解。
   (b) (3%) 在 $15:00$ 公司裡有多少人得知消息？
   (c) (4%) 求在 $15:00$，y'' 的值。在此時八卦的傳播速度是增加還是趨緩？

Solution:

(a) Let t be t hours after $9:00$.

$$y'(t) = \alpha y(1-y), \text{ for some constant } \alpha > 0. \text{ (2pts)} \quad y(0) = \frac{10}{400} = \frac{1}{40}. \text{ (1pt)}$$

If $y \neq 0$, $y \neq 1$, then $\dfrac{y'}{y(1-y)} = \alpha \Rightarrow \ln\left|\dfrac{y}{1-y}\right| = \alpha t + C$ (2pts)

$\because 0 < y(t) < 1 \therefore \dfrac{y(t)}{1-y(t)} = \kappa e^{\alpha t}$. (1pt)

By $y(0) = \dfrac{1}{40}$, we know that $\kappa = \dfrac{1}{39}$ (1pt)

$$\begin{aligned} y(t) &= \frac{1}{1+39e^{-\alpha t}}. \text{ At } 12:00, t = 12-9 = 3, \\ y(3) &= \frac{100}{400} = \frac{1}{1+39e^{-3\alpha}} \Rightarrow \alpha = \frac{1}{3}\ln 13 \text{or} e^{\alpha} = (13)^{\frac{1}{3}}. \text{(1pt)} \\ y(t) &= \frac{1}{1+39(13)^{-\frac{t}{3}}} \quad (2pts) \end{aligned}$$

(b) At $15:00$, $t = 15-9 = 6$, $y(6) = \dfrac{1}{1+39(13)^{-2}} = \dfrac{13}{16}$. (2pts). $400 \times \dfrac{13}{16} = 325$ (1pt) Ans: 325人

(c) $y'' = \alpha[y'(1-y) - yy'] = \alpha y'(1-2y) = \alpha^2 y(1-y)(1-2y)$ (2pts)
where $y = \dfrac{13}{16}$, $y'' = \alpha^2 \times \dfrac{13}{16} \times \dfrac{3}{16} \times \left(-\dfrac{10}{16}\right) = \dfrac{-5\times 13}{2^{11}\times 3}(\ln 13)^2$. (1pt 不必化簡)
$\because y'' < 0 \therefore y'$ is decreasing at 15:00. (1pt) Hence y' is decreasing at $15:00$.

⁎ 算 y'' 少 α^2 共扣 1 分。

⁎ y'' 算錯，但是由 $y'' < 0$ 推出 y' 趨緩，可以得最後部分的 1pt.

2. (12%) 解微分方程

$$\begin{cases} y'(t) = \dfrac{ty^2(t) - 3y^2(t)}{t^3}, & t > 0 \\ y(1) = -2. \end{cases}$$

Solution:

Write $y'(t) = y^2(t)\dfrac{t-3}{t^3}$. (2%) Separating variables gives

$$\int \frac{dy}{y^2} = \int \frac{t-3}{t^3} dt. \text{ (4\%)}$$

Then we have

$$-\frac{1}{y} = -\frac{1}{t} + \frac{3}{2t^2} + C. \text{ (3\%)}$$

On the other hand, $y(1) = -2$ implies that $C = 0$. (2 %) So we have

$$y(t) = \frac{2t^2}{2t-3}. \text{ (1\%)}$$

3. (12%) 解微分方程

$$\begin{cases} (t+1)y'(t) - 2(t^2+t)y(t) = \dfrac{e^{t^2}}{t+1}, \quad t > -1 \\ y(0) = 5. \end{cases}$$

Solution:

We write the equation in standard form

$$y'(t) - 2ty(t) = \frac{e^{t^2}}{(t+1)^2}. \ (3\%)$$

Let $u(t) = e^{\int -2t dt} = e^{-t^2}$. (3 %) Then

$$y(t) = e^{t^2} \int e^{-t^2} \cdot \frac{e^{t^2}}{(t+1)^2} dt = e^{t^2} \left(\frac{-1}{t+1} + C \right). \ (3\%)$$

On the other hand, $y(0) = 5 \Rightarrow 5 = -1 + C \Rightarrow C = 6$. (2 %) So

$$y(t) = 6e^{t^2} - \frac{e^{t^2}}{t+1}. \ (1\%)$$

4. (16%) 令 $y(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3 + \cdots$ 是微分方程

$$\begin{cases} \dfrac{dy}{dt} = y - t, \\ y(0) = 2. \end{cases}$$

以泰勒方法所得之解。

(a) (6%) 求出 a_0, a_1, a_2, a_3 的值。

(b) (4%) 使用此泰勒多項式 $a_0 + a_1 t + a_2 t^2 + a_3 t^3$ 求出 $y(0.2)$ 的近似值，四捨五入到小數點下第三位。

(c) (6%) 令 $(t_0, y_0) = (0, 2)$ 和 $h = \Delta t = 0.1$. 使用歐拉法求出 y_1 和 y_2.

Solution:

(a) Write $y(t) = [a_0, a_1, a_2, a_3, \cdots]$. By the differential equation, we have

$$[a_1, 2a_2, 3a_3, 4a_4, \cdots] = [a_0, a_1 - 1, a_2, a_3, \cdots].$$

Therefore,

$$\begin{aligned} a_1 &= a_0, \\ a_2 &= \frac{a_1 - 1}{2} = \frac{a_0 - 1}{2!}, \\ a_3 &= \frac{a_2}{3} = \frac{a_0 - 1}{3!}, \\ &\cdots, \end{aligned}$$

i.e.,

$$y(t) = a_0 + a_0 t + \frac{a_0 - 1}{2} t^2 + \frac{a_0 - 1}{3!} t^3 + \cdots. \ (2\%)$$

Since $y(0) = 2$, we have $a_0 = 2$. Therefore,

$$a_0 = 2, \quad a_1 = 2, \quad a_2 = \frac{1}{2}, \quad a_3 = \frac{1}{6}. \ (4\%)$$

(b) $2 + 2(0.2) + \frac{1}{2}(0.2)^2 + \frac{1}{6}(0.2)^3 \approx 2.421.$ (4%)

(c) Let $h = 0.1$ and $f(t, y) = y - t$. Then $t_1 = t_0 + h = 0.1$. (2 %) We have

$$y_1 = f(t_0, y_0)h + y_0 = f(0, 2) \cdot 0.1 + 2 = 2.2 \ (2\%)$$

and

$$\begin{aligned} y_2 = f(t_1, y_1)h + y_1 &= f(0.1, 2.2) \cdot 0.1 + 2.2 \\ &= 2.1 \cdot 0.1 + 2.2 = 2.41. \ (2\%) \end{aligned}$$

5. (14%) 已知 $\int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}$，求以下的積分。

(a) (7%) $\int_{-\infty}^{\infty} x^2 e^{-x^2}\,dx.$

(b) (7%) $\int_{-\infty}^{\infty} e^{-\frac{(x-1)^2}{4}}\,dx.$

Solution:

這一題「不需要」使用嚴格的瑕積分定義來計算。

(a)

$$\begin{aligned}\int_{-\infty}^{\infty} x^2 e^{-x^2}\,dx &= \int_{-\infty}^{\infty} x\,d\left(-\frac{1}{2}e^{-x^2}\right) \text{ (2pts: 使用 integration by parts，決定 } u,\, dv)\\ &= x\cdot\left(-\frac{1}{2}e^{-x^2}\right)\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty}\left(-\frac{1}{2}e^{-x^2}\right)dx \text{ (3pts: integration by parts)}\\ &= \frac{1}{2}\int_{-\infty}^{\infty} e^{-x^2}\,dx = \frac{\sqrt{\pi}}{2} \quad \left(\text{1pt: } x\left(-\frac{1}{2}e^{-x^2}\right)\Big|_{-\infty}^{\infty} = 0, \text{ 1pt: } \frac{1}{2}\int_{-\infty}^{\infty} e^{-x^2}\,dx = \frac{\sqrt{\pi}}{2}\right)\end{aligned}$$

(b) $\int_{-\infty}^{\infty} e^{-\frac{(x-1)^2}{4}}\,dx \overset{\text{Let } u=\frac{x-1}{2}\ du=\frac{1}{2}dx}{\underset{\text{3pts}}{=\!=\!=\!=\!=}} \int_{-\infty}^{\infty} e^{-u^2}\cdot 2du = 2\sqrt{\pi}$

(另外4分： 1pt for 上下界，1pt for e^{-u^2}, 1pt for dx=2du, 1pt for answer)

6. (17%)

(a) (7%) 決定常數 α 使得

$$f_X(t) = \begin{cases} \alpha t^{-\frac{1}{2}} e^{-t}, & t > 0 \\ 0, & t \le 0 \end{cases}.$$

為一機率密度函數。

(b) (10%) 對此隨機變數 $X \sim f_X$, 試求其期望值 $\mathbf{E}(X)$ 和變異數 $\mathbf{Var}(X)$.

Solution:

(a) We should have

$$\begin{aligned} 1 &= \int_0^\infty \alpha t^{-1/2} e^{-t}\, dt \quad (2\%) \\ &= \alpha \Gamma(1/2) \quad (2\%) \\ &= \sqrt{\pi}\alpha. \quad (3\%) \end{aligned}$$

Hence $\alpha = \dfrac{1}{\sqrt{\pi}}$.

(b) We have

$$\begin{aligned} \mathrm{E}(X) &= \frac{1}{\sqrt{\pi}} \int_0^\infty t^{1/2} e^{-t}\, dt \quad (1\%) \\ &= \frac{\Gamma(3/2)}{\sqrt{\pi}} \quad (2\%) \\ &= \frac{1}{2} \quad (2\%) \end{aligned}$$

and

$$\begin{aligned} \mathrm{E}(X^2) &= \frac{1}{\sqrt{\pi}} \int_0^\infty t^{3/2} e^{-t}\, dt \quad (1\%) \\ &= \frac{\Gamma(5/2)}{\sqrt{\pi}} = \frac{3}{2} \cdot \frac{1}{2} \quad (1\%) \\ &= \frac{3}{4} \quad (2\%). \end{aligned}$$

Thus

$$\mathrm{Var}(X) = \mathrm{E}(X^2) - \mathrm{E}(X)^2 = \frac{1}{2} \quad (1\%).$$

7. (12%) $f_X(t) = \frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{(t-\mu)^2}{2\sigma^2}}$, $t \in \mathbb{R}$ 是隨機變數 X 的機率密度函數，其中 μ, σ 是常數，$\sigma > 0$. 令隨機變數 $Y = \frac{X-\mu}{\sigma}$. 依照下述方法求出 Y的機率密度函數$f_Y(s)$.

(a) (3%) 說明 $\mathbf{P}(Y \le s) = \mathbf{P}(X \le \mu + \sigma s)$

(b) (3%) 上述所對應的積分等式為 $\int_{-\infty}^{s} f_Y(y)\,dy =$ ________________.

(c) (6%) 假設可以使用微積分基本定理，求出 $f_Y(s)$.

Solution:

(a) 因為

$$\mathbf{P}(Y \le s) = \mathbf{P}\left(\frac{X-\mu}{\sigma} \le s\right) = \mathbf{P}(X \le \mu + \sigma s)$$

(b) $\int_{-\infty}^{s} f_Y(y)\,dy =$

$$\int_{-\infty}^{\mu+\sigma s} \frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{(t-\mu)^2}{2\sigma^2}}\,dt \quad \text{或} \quad \int_{-\infty}^{\mu+\sigma s} f_X(t)\,dt$$

(c)

$$f_Y(s) = \left(\int_{-\infty}^{s} f_Y(y)\,dy\right)' = \left(\int_{-\infty}^{\mu+\sigma s} \frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{(t-\mu)^2}{2\sigma^2}}\,dt\right)' = \frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{s^2}{2}} \cdot \sigma = \frac{1}{\sqrt{2\pi}}e^{-\frac{s^2}{2}}$$

評分建議**.**
(a) 就是 0 或 3 分。
(b) 就是 0 或 3 分。
(c) 知道如何運用微積分基本定理給 3 分，算出正確答案給 3 分。若不會計算或蒜得很離譜，但是會敘述微積分基本定理如何套用在這類微分問題給基本分數 2 分。