1082 Calculus A 01-10, Mode 01-02 Makeup Midterm

1. (15 pts) Consider the function

$$f(x,y) = \begin{cases} \dfrac{\sin(xy)}{\sqrt{x^2+y^2}} & \text{,if } (x,y) \neq (0,0), \\ 0 & \text{,if } (x,y) = (0,0). \end{cases}$$

   (a) (3 pts) Is $f(x,y)$ continuous at $(0,0)$?

   (b) (4 pts) Find $f_x(0,0), f_y(0,0)$, and $f_x(x,y), f_y(x,y)$ for $(x,y) \neq (0,0)$.

   (c) (2 pts) Write down the linear approximation of $f(x,y)$ at $(0,0)$, $L(x,y)$.

   (d) (4 pts) Compute $\displaystyle\lim_{(x,y)\to(0,0)} \frac{f(x,y) - L(x,y)}{\sqrt{x^2+y^2}}$ if it exists.

   (e) (2 pts) Is $f(x,y)$ differentiable at $(0,0)$?

Solution:

(a) We consider polar coordinate, let $x = r\cos\theta$ and $y = r\sin\theta$, then

$$f(x,y) = f(r,\theta) = \begin{cases} \dfrac{\sin(r^2\cos\theta\sin\theta)}{\sqrt{r^2}} & \text{,if } r \neq 0, \\ 0 & \text{,if } r = 0. \end{cases}$$

By definition of continuous, we can consider the countinuity of $f(x,y)$ at $(x,y) = (0,0)$ by taking $r \to 0^+$. $\displaystyle\lim_{r\to0^+} f(r,\theta) = \lim_{r\to0^+} \frac{\sin(r^2\cos\theta\sin\theta)}{r} = \lim_{r\to0^+} \frac{\sin(r^2\cos\theta\sin\theta)}{r^2\cos\theta\sin\theta} \cdot r\cos\theta\sin\theta$.
Since both $\displaystyle\lim_{r\to0^+} \frac{\sin(r^2\cos\theta\sin\theta)}{r^2\cos\theta\sin\theta}$ and $\displaystyle\lim_{r\to0^+} r\cos\theta\sin\theta$ are exist with $\displaystyle\lim_{r\to0^+} \frac{\sin(r^2\cos\theta\sin\theta)}{r^2\cos\theta\sin\theta} = 1$ and $\displaystyle\lim_{r\to0^+} r\cos\theta\sin\theta = 0$, then

$$\lim_{r\to0^+} \frac{\sin(r^2\cos\theta\sin\theta)}{r^2\cos\theta\sin\theta} \cdot r\cos\theta\sin\theta = \lim_{r\to0^+} \frac{\sin(r^2\cos\theta\sin\theta)}{r^2\cos\theta\sin\theta} \cdot \lim_{r\to0^+} r\cos\theta\sin\theta = 1 \cdot 0 = 0.$$

That is $\displaystyle\lim_{(x,y)\to(0,0)} f(x,y) = f(0,0)$, so we have $f(x,y)$ is continuous at $(0,0)$.

(b) By definition of partial derivative at $(0,0)$.

$$\begin{cases} f_x(0,0) = \displaystyle\lim_{h\to0} \frac{f(h,0) - f(0,0)}{h} = \lim_{h\to0} \frac{\frac{\sin(h\cdot0)}{\sqrt{h^2}} - 0}{h} = 0 \\ f_y(0,0) = \displaystyle\lim_{k\to0} \frac{f(0,k) - f(0,0)}{k} = \lim_{k\to0} \frac{\frac{\sin(0\cdot k)}{\sqrt{k^2}} - 0}{k} = 0 \end{cases}$$

and we can derivative for $(x,y) \neq (0,0)$ since $f(0,0)$ is differentiable except $(x,y) = (0,0)$.

$$\begin{cases} f_x(x,y) = \dfrac{\cos(xy)\cdot y \cdot \sqrt{x^2+y^2} - \sin(xy)\cdot \frac{1}{2}\frac{2x}{\sqrt{x^2+y^2}}}{(\sqrt{x^2+y^2})^2} = \dfrac{y(x^2+y^2)\cos(xy) - x\sin(xy)}{(x^2+y^2)^{\frac{3}{2}}} \\ f_y(x,y) = \dfrac{\cos(xy)\cdot x \cdot \sqrt{x^2+y^2} - \sin(xy)\cdot \frac{1}{2}\frac{2y}{\sqrt{x^2+y^2}}}{(\sqrt{x^2+y^2})^2} = \dfrac{x(x^2+y^2)\cos(xy) - y\sin(xy)}{(x^2+y^2)^{\frac{3}{2}}} \end{cases}$$

(c) Linear approximation of $f(x,y)$ at $(0,0)$ is

$$\begin{aligned} L(x,y) &= f(0,0) + f_x(0,0)x + f_y(0,0)y \\ &= 0 + 0 + 0 = 0 \end{aligned}$$

(d)

$$\lim_{(x,y)\to(0,0)} \frac{f(x,y) - L(x,y)}{\sqrt{x^2+y^2}} = \lim_{(x,y)\to(0,0)} \frac{\sin(xy)}{x^2+y^2}.$$

Consider two straight line $x = y$ and $x = -y$ then

$$\begin{cases} \lim\limits_{(x,y)\to(0,0)} \dfrac{\sin(xy)}{x^2+y^2} = \lim\limits_{x\to 0} \dfrac{\sin(x^2)}{2x^2} = \dfrac{1}{2}, & \text{if } \ x = y \\ \lim\limits_{(x,y)\to(0,0)} \dfrac{\sin(xy)}{x^2+y^2} = \lim\limits_{x\to 0} \dfrac{\sin(-x^2)}{2x^2} = -\dfrac{1}{2}, & \text{if } \ x = -y \end{cases}$$

(d)

$$\lim_{(x,y)\to(0,0)} \frac{f(x,y) - L(x,y)}{\sqrt{x^2+y^2}} = \lim_{(x,y)\to(0,0)} \frac{\sin(xy)}{x^2+y^2}.$$

That is the limit $\lim\limits_{(x,y)\to(0,0)} \dfrac{f(x,y) - L(x,y)}{\sqrt{x^2+y^2}}$ does not exist.

(e) No, $f(x,y)$ is not differentiable at $(0,0)$ since $\lim\limits_{(x,y)\to(0,0)} \dfrac{f(x,y) - L(x,y)}{\sqrt{x^2+y^2}}$ does not exist.

2. (15 pts) Consider the level surface S defined by

$$S = \{(x, y, z) \in \mathbb{R}^3 : z^5 - xz^4 + yz^3 = 1\}.$$

This level surface defines $z = z(x, y)$ implicitly as a differentiable function of x and y.

(a) (4 pts) Find an equation of the tangent plane at the point $P(0, 0, z(0, 0).)$

(b) (4 pts) Use linear approximation at P to estimate the value $z(0.02, -0.03)$.

(c) (5 pts) Find $\left.\dfrac{\partial^2 z}{\partial x \partial y}\right|_{(x,y)=(0,0)}$.

(d) (2 pts) Find the directional derivative $D_u z(0, 0)$ along the direction $\vec{u}$ parallel to the vector $(-3, 4)$.

Solution:

(a) Let $F(x, y, z) = z^5 - xz^4 + yz^3 - 1$, then $S = \{(x, y, z) \in \mathbb{R}^3 : F(x, y, z) = 0\}$, then the normal vector of tangent plane at each point (x, y, z) is $\nabla F(x, y, z) = (-z^4, z^3, 5z^4 - 4xz^3 + 3yz^2)$, that is the normal vector at $(0, 0, z(0, 0)) = (0, 0, 1)$ is $(-1, 1, 5)$. Then the tangent plane at the point $P(0, 0, 1)$ is $-(x - 0) + (y - 0) + 5(z - 1) = 0$.

(b) Linear approximate at P is $L(x, y) = z(0, 0) + z_x(0, 0)(x - 0) + z_y(0, 0)(y - 0) = z(0, 0) - \dfrac{F_x(0,0,0)}{F_z(0,0,0)}(x - 0) - \dfrac{F_y(0,0,0)}{F_z(0,0,0)}(y - 0) \Rightarrow z(0.02, -0.03) = 1 + \dfrac{1}{5} \cdot 0.02 - \dfrac{1}{5} \cdot (-0.03) = 1.01$
Actually, by tangent plane at $P(0, 0, 1)$, $-(0.02 - 0) + (-0.03 - 0) + 5(z(0.02, -0.03) - 1) = 0 \Rightarrow 5(z(0.02, -0.03) - 1) = 0.05 \Rightarrow z(0.02, -0.03) = 1 + \dfrac{0.05}{5} = 1.01$.

(c) Since $\dfrac{\partial z}{\partial y} = -\dfrac{F_y(x, y, z)}{F_z(x, y, z)} = -\dfrac{z^3}{5z^4 - 4xz^3 + 3yz^2} = \dfrac{-z}{5z^2 - 4xz + 3y}$

$$\begin{aligned}
\Rightarrow \left.\frac{\partial^2 z}{\partial x \partial y}\right|_{(x,y)=(0,0)} &= \left.\frac{\partial}{\partial x}\left(\frac{-z}{5z^2 - 4xz + 3y}\right)\right|_{(x,y)=(0,0)} \\
&= \left.\frac{-z_x(5z^2 - 4xz + 3y) + z(10zz_x - 4(z + xz_x))}{(5z^2 - 4xz + 3y)^2}\right|_{(x,y)=(0,0)} \\
&= \frac{-\frac{1}{5} \cdot 5 + 1 \cdot \left(10 \cdot 1 \cdot \frac{1}{5} - 4\right)}{5^2} = \frac{-1 + (-2)}{25} = -\frac{3}{25}
\end{aligned}$$

(d) $D_{\vec{u}} z(0, 0) = \nabla z(0, 0) \cdot \vec{u} = \left(\dfrac{1}{5}, -\dfrac{1}{5}\right) \cdot \pm\left(\dfrac{-3}{5}, \dfrac{4}{5}\right) = \pm\dfrac{7}{25}$

3. (14 pts) Let $f(x,y) = y(x+1)^2$.

(a) (6 pts) Find and classify critical point(s) of $f(x,y)$.

(b) (8 pts) Find the extreme values of $f(x,y)$ on the region $R = \{(x,y)|x^2-1 \le y \le 3\}$.

Solution:

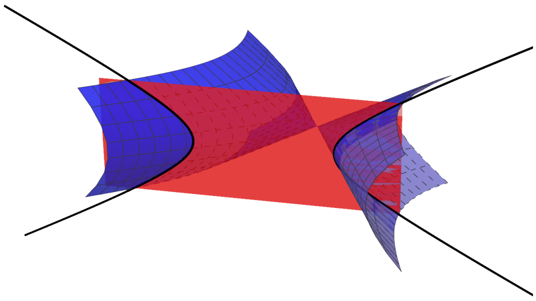
(a) $\nabla f(x,y) = (2y(x+1),(x+1)^2) = (0,0)$ iff $(x,y) = (-1,y)$

$$\Rightarrow \begin{cases} f_{xx} = 2y \\ f_{xy} = 2(x+1) = f_{yx} \\ f_{yy} = 0 \end{cases} \Rightarrow D|_{(x,y)=(-1,y)} = f_{xx}f_{yy} - f_{xy}^2|_{(x,y)=(-1,y)} = 0 \quad \forall y \in \mathbb{R}.$$

If $y > 0$, $f(x,y) \ge 0$ for all $x \ne -1$,then $(-1,y)$ is local minimum. With same argument for $y < 0$, $(-1,y)$ is local maximun. The last one is $(-1,0)$ is a saddle point.

(b) $R = \{(x,y)|x^2-1 \le y \le 3\}$.
First, we have the local minimum in the region equal to 0 when $(x,y) = (-1,y)$ and $y > 0$.
Second, we consider the boundary $x^2-1 \le y \le 3 \Rightarrow x^2-1 = 3$ with $-2 \le x \le 2$ and $y = 3$ then we have the local maximum when $(x,y) = (2,3)$ is equal to 27.
The last, $y = x^2-1 \Rightarrow f(x,y) = f(x,x^2-1) = (x^2-1)(x+1)^2 = g(x)$ with $-2 \le x \le 2$, then $g'(x) = 2x(x+1)^2+2(x^2-1)(x+1) = 2(x+1)(x^2+x+x^2-1) = 2(x+1)(2x^2+x-1) = 2(x+1)^2(2x-1) = 0$ for $x = -1, \frac{1}{2} \Rightarrow (x,y) = \left(\frac{1}{2},-\frac{3}{4}\right)$ has local minimum is equal to $-\frac{27}{16}$.

4. (10 pts) Let C be the hyperbola formed by the intersection of the cone $4x^2 = y^2 + 3z^2$ and the plane $x + y = 6$. Find the maximum and the minimum distance between the origin and the point on C (if exist) by the method of Lagrange multipliers.



Solution:

That is to find the extreme of $d(x, y, z) = x^2 + y^2 + z^2$ with the restriction $x + y = 6$ and $4x^2 = y^2 + 3z^2$
By Lagrange Multipliers

$$\begin{cases} 2x = \lambda + \mu 8x \\ 2y = \lambda + \mu(-2y) \\ 2z = 0 + \mu(-6z) \end{cases} \Rightarrow \begin{cases} \lambda = (2 - 8\mu)x = (2 + 2\mu)y \\ \mu = -\dfrac{1}{3} \quad \text{or} \quad z = 0 \end{cases} \Rightarrow \begin{cases} \lambda = \dfrac{14}{3}x = \dfrac{4}{3}y \\ \text{or} \\ \lambda = (2 - 8\mu)x = (2 + 2\mu)y, z = 0 \end{cases}$$

(i) If $\lambda = \dfrac{14}{3}x = \dfrac{4}{3}y$, $x + y = 6$ and $4x^2 = y^2 + 3z^2$
$\Rightarrow \dfrac{3}{14}\lambda + \dfrac{3}{4}\lambda = 6 \Rightarrow \dfrac{27}{28}\lambda = 6 \Rightarrow \lambda = \dfrac{56}{9} \Rightarrow (x, y) = \left(\dfrac{4}{3}, \dfrac{14}{3}\right) \Rightarrow (x, y, z)$ not exists in $\mathbb{R}^3$.

(ii) If $\lambda = (2 - 8\mu)x = (2 + 2\mu)y, z = 0$, $x + y = 6$ and $4x^2 = y^2 + 3z^2$
$\Rightarrow 4x^2 = y^2 \Rightarrow 2x = \pm y$ and $x + y = 6 \Rightarrow (x, y) = (2, 4)$ or $(-6, 12)$ Since there is no maximun distance between origin and the point on hyperbola C, we have the minimun at $(2, 4, 0)$ is equal to $\sqrt{20}$.

5. (12 pts)

(a) Compute $\int_0^1 \int_0^{\cos^{-1} y} \sin x(1+\sin^2 x)^{\frac{1}{3}} \mathrm{d}x\mathrm{d}y$.

(b) Let $f(x) = \int_x^1 e^{-t^2} \mathrm{d}t$. Find the average value of f on the interval $[0,1]$.

Solution:

(a)

$$\begin{aligned}\int_0^1 \int_0^{\cos^{-1} y} \sin x(1+\sin^2 x)^{\frac{1}{3}} dxdy &= \int_0^{\frac{\pi}{2}} \int_0^{\cos x} \sin x(1+\sin^2 x)^{\frac{1}{3}} dydx = \int_0^{\frac{\pi}{2}} \sin x(1+\sin^2 x)^{\frac{1}{3}} \cos x dx \\ &= \frac{1}{2}(1+\sin^2 x)^{\frac{4}{3}} \cdot \frac{3}{4}\Big|_0^{\frac{\pi}{2}} = \frac{3\cdot 2^{\frac{4}{3}}}{8} - \frac{3}{8}\end{aligned}$$

(b) Average value of $f = \dfrac{\int_0^1 f dx}{1-0} = \int_0^1 \int_x^1 e^{-t^2} dtdx$

$$\Rightarrow \int_0^1 \int_x^1 e^{-t^2} dtdx = \int_0^1 \int_0^t e^{-t^2} dxdt = \int_0^1 te^{-t^2} dt = \frac{-1}{2} e^{-t^2}\Big|_0^1 = \frac{1}{2}(1-e^{-1})$$

6. (12 pts) Find the center of mass of a lamina

$$D = \left\{(x,y) \in \mathbf{R}^2 \middle| \frac{(x-2)^2}{4} + y^2 \le 1 \text{ and } x \le 2\right\}$$

whose density function at any point is proportional to the square of its distance from the line $x = 2$.

Solution:

Assume the coordinate of center of mass is $(\bar{x}, \bar{y})$, then obvious $\bar{y} = 0$.

$$\Rightarrow \bar{x} \cdot \iint_D C(x^2+y^2)dA = \iint_D Cx(x^2+y^2)dA \Rightarrow \bar{x} = \frac{\iint_D x(x^2+y^2)dA}{\iint_D (x^2+y^2)dA}$$

$$\begin{cases} \displaystyle\iint_D (2-x)^2 dA = \int_0^2 \int_{-\sqrt{1-\frac{(x-2)^2}{4}}}^{\sqrt{1-\frac{(x-2)^2}{4}}} (2-x)^2 dy dx = \int_0^2 2\sqrt{1-\frac{(x-2)^2}{4}}(2-x)^2 dx \\ \displaystyle\iint_D x(2-x)^2 dA = \int_0^2 \int_{-\sqrt{1-\frac{(x-2)^2}{4}}}^{\sqrt{1-\frac{(x-2)^2}{4}}} x(2-x)^2 dy dx = \int_0^2 2\sqrt{1-\frac{(x-2)^2}{4}}x(2-x)^2 dx \end{cases}$$

Let $t = \frac{x-2}{2} \Rightarrow dx = 2dt$

$$\Rightarrow \begin{cases} \displaystyle\int_0^2 2\sqrt{1-\frac{(x-2)^2}{4}}(2-x)^2 dx = \int_{-1}^0 2\sqrt{1-t^2}4t^2 \cdot 2dt \\ \displaystyle\int_0^2 2\sqrt{1-\frac{(x-2)^2}{4}}x(2-x)^2 dx = \int_{-1}^0 2\sqrt{1-t^2} \cdot (2t+2) \cdot 4t^2 \cdot 2dt \end{cases}$$

Let $t = \cos\theta \Rightarrow dt = -\sin\theta d\theta$

$$\Rightarrow \begin{cases} \displaystyle -\int_{-1}^0 2\sqrt{1-t^2}4t^2 \cdot 2dt = \int_{\frac{\pi}{2}}^{\pi} 16\sin\theta\cos^2\theta\sin\theta d\theta = 16\int_{\frac{\pi}{2}}^{\pi} \sin^2\theta\cos^2\theta d\theta \\ \displaystyle -\int_{-1}^0 2\sqrt{1-t^2} \cdot (2t+2) \cdot 4t^2 \cdot 2dt = \int_{\frac{\pi}{2}}^{\pi} 32\sin\theta\cos^2\theta\sin\theta(\cos\theta+1)d\theta = 32\int_{\frac{\pi}{2}}^{\pi} \sin^2\theta\cos^3\theta + \sin^2\theta\cos^2\theta d\theta \end{cases}$$

$$\Rightarrow \begin{cases} \displaystyle 16\int_{\frac{\pi}{2}}^{\pi} \sin^2\theta\cos^2\theta d\theta = 2\int_{\frac{\pi}{2}}^{\pi} \sin^2 2\theta d2\theta = 2\int_{\frac{\pi}{2}}^{\pi} \cos^2 2\theta d2\theta = 2\int_{\frac{\pi}{2}}^{\pi} \frac{\sin^2 2\theta + \cos^2 2\theta}{2} d2\theta = 2\pi - \pi = \pi \\ \displaystyle 32\int_{\frac{\pi}{2}}^{\pi} \sin^2\theta\cos^3\theta d\theta = 32\int_{\frac{\pi}{2}}^{\pi} \sin^2\theta(1-\sin^2\theta)\cos\theta d\theta = 32\left(\frac{1}{3}\sin^3\theta - \frac{1}{5}\sin^5\theta\right)_{\frac{\pi}{2}}^{\pi} = -\frac{64}{15} \end{cases}$$

$$\Rightarrow \iint_D (2-x)^2 dA = \pi \quad \text{and} \quad \iint_D x(2-x)^2 dA = -\frac{64}{15} + 2\pi$$

$$\therefore \bar{x} = \frac{2\pi - \frac{64}{15}}{\pi} \Rightarrow \text{center of mass : } (\bar{x}, \bar{y}) = \left(2 - \frac{64}{15\pi}, 0\right)$$

7. (12 pts) Use spherical coordinates to evaluate $\int_0^{\sqrt{3}} \int_{-\sqrt{3-y^2}}^{\sqrt{3-y^2}} \int_1^{\sqrt{4-x^2-y^2}} \frac{1}{\sqrt{x^2+y^2+z^2}} \mathrm{d}z\mathrm{d}x\mathrm{d}y.$

Solution:

By sperical coordinates, let $\begin{cases} x = \rho\cos\theta\sin\phi \\ y = \rho\sin\theta\sin\phi \\ z = \rho\cos\phi \end{cases}$, where $0 \le \rho \le 2$ $0 \le \phi \le \pi$ and $0 \le \theta \le 2\pi$.

$$\begin{aligned} \Rightarrow \int_0^{\sqrt{3}} \int_{-\sqrt{3-y^2}}^{\sqrt{3-y^2}} \int_1^{\sqrt{4-x^2-y^2}} \frac{1}{\sqrt{x^2+y^2+z^2}} dzdxdy &= \int_0^{\pi} \int_0^{\frac{\pi}{3}} \int_{\sec\phi}^{2} \frac{\rho^2\sin\phi}{\rho} d\rho d\phi d\theta = \frac{\pi}{2}\int_0^{\frac{\pi}{3}} (4-\sec^2\phi)\sin\phi d\phi \\ &= \frac{\pi}{2}\int_0^{\frac{\pi}{3}} 4\sin\phi - \sec\phi\tan\phi d\phi = \frac{\pi}{2}(-4\cos\phi - \sec\phi)\Big|_0^{\frac{\pi}{3}} = \frac{\pi}{2} \end{aligned}$$

8. (10 pts) Compute $\iint_R \ln(x^2y+x)\mathrm{d}A$, where R is the region bounded by curves $xy=1$, $xy=3$, $x=1$ and $x=e$.

Solution:

By change variables, let $\begin{cases} u = xy, 1 \le u \le 3 \\ v = x, 1 \le v \le e \end{cases} \Rightarrow |J| = \left|\dfrac{\partial(x,y)}{\partial(u,v)}\right| = \left| \begin{vmatrix} 0 & 1 \\ \frac{1}{v} & \frac{-u}{v^2} \end{vmatrix} \right| = \dfrac{1}{v}$

$$\begin{aligned}
\Rightarrow \iint_R \ln(x^2y+x)dA &= \int_1^e \int_1^3 \ln(uv+v)|J|dudv = \int_1^e \int_1^3 \ln[(u+1)v]\frac{1}{v}dudv \\
&= \int_1^e \int_1^3 \frac{\ln(u+1)+\ln v}{v}dudv \\
&= \int_1^e [(u+1)\ln(u+1)-(u+1)]_1^3 \frac{1}{v} + (3-1)\frac{\ln v}{v}dv \\
&= \ln e[(3+1)\ln(3+1)-(3+1)-2\ln 2+2] + \frac{(3-1)}{2}(\ln e)^2 \\
&= 8\ln 2 - 4 - 2ln2 + 2 + 1 = 6\ln 2 - 1
\end{aligned}$$