1. (15%) Determine whether the series is absolutely convergent, conditionally convergent, or divergent. Please state the tests which you use.

   (a) (5%) $\sum_{n=1}^{\infty}(-1)^n(\sqrt{n^2+1}-\sqrt{n^2-1})$.

   (b) (5%) $\sum_{n=1}^{\infty}\frac{\cos(n)}{n\sqrt{n}}$.

   (c) (5%) $\sum_{n=1}^{\infty}\frac{n^n}{2^n n!}$.

Solution:

(a) Since $\lim_{n\to\infty}|(-1)(\sqrt{n^2+1}-\sqrt{n^2-1})| = \lim_{n\to\infty}\frac{2}{\sqrt{n^2+1}+\sqrt{n^2-1}} = 0$ and

$$\sqrt{n^2+1}+\sqrt{n^2-1} < \sqrt{(n+1)^2+1}+\sqrt{(n+1)^2-1} \Rightarrow \frac{2}{\sqrt{n^2+1}+\sqrt{n^2-1}} > \frac{2}{\sqrt{(n+1)^2+1}+\sqrt{(n+1)^2-1}}$$

by alternating series test, we have $\sum_{n=1}^{\infty}(-1)^n(\sqrt{n^2+1}-\sqrt{n^2-1})$ converge.

But $(\sqrt{n^2+1}+\sqrt{n^2-1})^2 \le (n+1)+n \Rightarrow \frac{2}{\sqrt{n^2+1}+\sqrt{n^2-1}} > \frac{2}{2n+1}$ by p-series $\sum_{n=1}^{\infty}\frac{2}{2n+1}$ diverges, so $\sum_{n=1}^{\infty}|(-1)^n(\sqrt{n^2+1}-\sqrt{n^2-1})| = \sum_{n=1}\frac{2}{\sqrt{n^2+1}+\sqrt{n^2-1}}$ also diverges. That is $\sum_{n=1}^{\infty}(-1)^n(\sqrt{n^2+1}-\sqrt{n^2-1})$ is conditionally convergent.

(b) By p-series, $\sum_{n=1}^{\infty}\frac{1}{n^{\frac{3}{2}}}$ is convergent, then by comparison test, $\left|\frac{\frac{\cos(n)}{n\sqrt{n}}}{\frac{1}{n^{\frac{3}{2}}}}\right| < 1 \Rightarrow \sum_{n=1}^{\infty}\frac{\cos(n)}{n\sqrt{n}}$ is absolutely convergnet.

(c) By ratio test, $\lim_{n\to\infty}\left|\frac{\frac{(n+1)^{n+1}}{2^{n+1}(n+1)!}}{\frac{n^n}{2^n n!}}\right| = \lim_{n\to\infty}\left|\frac{(n+1)^{n+1}}{2(n+1)n^n}\right| = \lim_{n\to\infty}\frac{1}{2}\left(1+\frac{1}{n}\right)^n = \frac{e}{2} > 1.$

So $\sum_{n=1}^{\infty}\frac{n^n}{2^n n!}$ is divergent.

2. (14%) Find the interval of convergence of the power series and find the function represented by it.

(a) (7%) $\sum_{n=2}^{\infty}(-1)^n n(n-1)x^n$.

(b) (7%) $\sum_{n=0}^{\infty}\frac{1}{(n+2)n!}x^n$.

Solution:

(a) Consider $\frac{1}{1-x}=\sum_{n=0}^{\infty}(-x)^n=\sum_{n=0}^{\infty}(-1)^n x^n$ for $|x|<1$. Then

$$\frac{1}{(1-x)^2}=\frac{d}{dx}(\frac{1}{1-x})=\sum_{n=1}^{\infty}(-1)^n n x^{n-1} \quad -1<x<1,$$
$$\frac{2}{(1-x)^3}=\frac{d}{dx}[\frac{1}{(1-x)^2}]=\sum_{n=2}^{\infty}(-1)^n n(n-1)x^{n-2} \quad -1<x<1.$$

For $x=1$, the series $\sum_{n=2}^{\infty}(-1)^n n(n-1)$ diverges since $\lim_{n\to\infty}(-1)^n n(n-1)\neq 0$.
For $x=-1$, the series $\sum_{n=2}^{\infty}n(n-1)$ diverges since $\lim_{n\to\infty}n(n-1)\neq 0$.
Combining above results, we have

$$\frac{2}{(1-x)^3}=\sum_{n=2}^{\infty}(-1)^n n(n-1)x^{n-2}, \qquad -1<x<1$$

(b) Let $a_n=\frac{1}{(n+2)\cdot n!}x^n$. Since

$$\lim_{n\to\infty}|\frac{a_{n+1}}{a_n}|=\lim_{n\to\infty}|\frac{x^{n+1}}{(n+3)\cdot(n+1)!}\cdot\frac{(n+2)\cdot n!}{x^n}|=\lim_{n\to\infty}|\frac{(n+2)}{(n+1)(n+3)}||x|=0<1$$

for each real number x, by ratio test, the interval of convergence of $\sum_{n=0}^{\infty}\frac{1}{(n+2)\cdot n!}x^n$ is $\mathbb{R}$.

Consider $f(x)=\sum_{n=0}^{\infty}\frac{1}{(n+2)\cdot n!}x^n$. Then we have

$$x^2 f(x)=\sum_{n=0}^{\infty}\frac{1}{(n+2)\cdot n!}x^{n+2}$$
$$\frac{d}{dx}[x^2 f(x)]=\sum_{n=0}^{\infty}\frac{1}{(n+2)\cdot n!}\cdot(n+2)\cdot x^{n+1}=\sum_{n=0}^{\infty}\frac{1}{n!}x^{n+1}=xe^x$$
$$x^2 f(x)=\int_0^x te^t\,dt=xe^x-e^x+1$$
$$f(x)=\begin{cases}\frac{1}{x^2}(xe^x-e^x+1) & ,\text{if } x\neq 0,\\ \frac{1}{2} & ,\text{if } x=0.\end{cases}$$

P.S. We also accept the answer $f(x)=\frac{1}{x^2}(xe^x-e^x+1)$.

3. (15%)

(a) (4%) Find the Maclaurin series for $g(x) = x\int_0^x e^{-t^2}\,dt$.

(b) (3%) Find $g^{(2020)}(0)$.

(c) (5%) Find the Maclaurin series for $f(x) = \begin{cases} \dfrac{\tan^{-1} x}{x}, & \text{for } x \neq 0 \\ 1, & \text{for } x = 0 \end{cases}$.

(d) (3%) Find $\displaystyle\lim_{x\to 0} \frac{x^2 f(x) - g(x)}{x^6}$.

Solution:

(a)

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

$$e^{-t^2} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} t^{2n} = 1 - t^2 + \frac{t^4}{2!} - \frac{t^6}{3!} + \cdots$$

$$g(x) = x\int_0^x e^{-t^2}\,dt = x\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)n!} x^{2n+1}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)n!} x^{2n+2} = x^2 - \frac{x^4}{3} + \frac{x^6}{5\cdot 2!} - \frac{x^8}{7\cdot 3!} + \cdots$$

(b)

$$g^{(2020)}(0) = c_{2020}\cdot(2020)!$$

Here c_{2020} is the coefficient of x^{2020} in the Maclaurin series for $g(x)$. Take $2n+2 = 2020$, $n = 1009$

$$g^{(2020)}(0) = \frac{-1}{2019\cdot(1009)!}\cdot(2020)!$$

(c) Start with the Maclaurin series for $\tan^{-1} x$

$$\tan^{-1} x = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1} = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \cdots$$

$$f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n} = 1 - \frac{1}{3}x^2 + \frac{1}{5}x^4 - \frac{1}{7}x^6 + \cdots$$

(d)

$$\lim_{x\to 0} \frac{x^2 f(x) - g(x)}{x^6} = \frac{1}{5} - \frac{1}{10} = \frac{1}{10}$$

Grading scheme:

(a) (4%) 1 point to each process: (1) Starting function's Maclaurin series. (2) Plug in t^2. (3) Integrate. (4) Multiply.

(b) (3%) 1 point to formula, 1 point to finding correct n (depends on their answer from (a)), 1 point for final answer. [essentially, 1 point for each part of the answer]

(c) (5%) 3 points for $\tan^{-1} x$ and 2 points for getting to $f(x)$.

(d) (3%) Depends on the previous answers. 1 point for knowing the method (even if they have nothing in (a) and (c)), 1 point each for the coefficients (no points if these are not from (a) and (c)).

Alternatively, they can plug in the functions into (d) and use l'Hospital's Rule (3 points for correct answer, all or nothing).

4. (10%) Let $f(x) = \int_0^{3x} \sin(t^2) + t^2\cos(t^2)\,dt$.

(a) (5%) Find the Taylor series of $f(x)$ centered at $x = 0$.

(b) (5%) Now we approximate $f(0.1)$ with $T_7(0.1)$ where $T_7(x)$ is the degree 7 Taylor polynomial centered at $x = 0$. Estimate the error.

Solution:

(a) Power series

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}$$

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}$$

$$\sin(t^2) + t^2\cos(t^2) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} t^{4n+2} + t^2 \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} t^{4n} = \sum_{n=0}^{\infty} (-1)^n \left(\frac{1}{(2n+1)!} + \frac{1}{(2n)!} \right) t^{4n+2}$$

$$f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{4n+3} \left(\frac{1}{(2n+1)!} + \frac{1}{(2n)!} \right) (3x)^{4n+3}$$

(b) Error estimate

Since the series is alternating, the error estimate is given by

$$|f(0.1) - T_7(0.1)| < \frac{1}{11} \left(\frac{1}{5!} + \frac{1}{4!} \right) (0.3)^{11}$$

5. (8%) Let $\mathbf{F}(x,y,z) = e^{yz}\,\mathbf{i} + (xze^{yz} + z\cos y)\,\mathbf{j} + (xye^{yz} + \sin y)\,\mathbf{k}$.

(a) (4%) Compute $\operatorname{curl}\mathbf{F}$ and $\operatorname{div}\mathbf{F}$.

(b) (4%) Determine whether or not the vector field is conservative. If it is conservative, find a function f such that $\nabla f = \mathbf{F}$.

Solution:

(a) By definition, $\operatorname{curl}\mathbf{F} = \nabla \times \mathbf{F}$

$$\begin{aligned} &= \left(\frac{\partial}{\partial z}(xze^{yz} + z\cos y) - \frac{\partial}{\partial y}(xye^{yz} + \sin y)\right)\mathbf{i} + \left(\frac{\partial}{\partial x}(xye^{yz} + \sin y) - \frac{\partial}{\partial z}e^{yz}\right)\mathbf{j} \\ &\quad + \left(\frac{\partial}{\partial y}e^{yz} - \frac{\partial}{\partial x}(xze^{yz} + z\cos y)\right)\mathbf{k} \\ &= ((xe^{yz} + xyze^{yz} + \cos y) - (xe^{yz} + xyze^{yz} - \cos y))\,\mathbf{i} + (ye^{yz} - ye^{yz})\,\mathbf{j} + (ze^{yz} - ze^{yz})\,\mathbf{k} = 0\,\mathbf{i} + 0\,\mathbf{j} + 0\,\mathbf{k} \end{aligned}$$

and $\operatorname{div}\mathbf{F} = \nabla\cdot\mathbf{F} = \dfrac{\partial}{\partial x}e^{yz} + \dfrac{\partial}{\partial y}(xze^{yz} + z\cos y) + \dfrac{\partial}{\partial z}(xye^{yz} + \sin y) = x(y^2+z^2)e^{yz} - z\sin y$

(b) Yes, since $\operatorname{curl}\mathbf{F} = (0,0,0)$. That is $\nabla f = \mathbf{F}(x,y,z) = e^{yz}\,\mathbf{i} + (xze^{yz} + z\cos y)\,\mathbf{j} + (xye^{yz} + \sin y)\,\mathbf{k}$

$$\begin{cases} f_x = e^{yz} \\ f_y = xze^{yz} + z\cos y \\ f_z = xye^{yz} + \sin y \end{cases} \Rightarrow \begin{cases} f = xe^{yz} + g_1(y,z) \\ f = xe^{yz} + z\sin y + g_2(x,z) \\ f = xe^{yz} + z\sin y + g_3(x,y) \end{cases}$$

$f(x,y,z) = xe^{yz} + z\sin y + C$.

6. (11%) Let $\mathbf{F}(x,y) = \left(\dfrac{x^3+4xy^2-2y}{x^2+4y^2}\right)\mathbf{i} + \left(\dfrac{16y^3+4x^2y+2x}{x^2+4y^2}\right)\mathbf{j}$.

(a) (5%) Compute $\int_{C_1} \mathbf{F}\cdot d\mathbf{r}$, where C_1 is the ellipse $x^2+4y^2=1$ oriented counterclockwise.

(b) (2%) Is $\mathbf{F}$ conservative on $\mathbb{R}^2\backslash\{(0,0)\}$?

(c) (4%) Compute $\int_{C_2} \mathbf{F}\cdot d\mathbf{r}$, where C_2 is a square oriented counterclockwise with vertices $(10,0),(0,10),(-10,0)$ and $(0,-10)$.

Solution:

(a) For all points $P\in C_1$, $\mathbf{F}(P) = (x-2y)\,\mathbf{i} + (4y+2x)\,\mathbf{j}$.

Consider polar coordinate we have $\begin{cases} x = \cos\theta \\ y = \dfrac{1}{2}\sin\theta \end{cases}$ $\theta\in[0,2\pi)$. Then

$$\begin{aligned}
\int_{C_1}\mathbf{F}\cdot d\mathbf{r} &= \int_0^{2\pi}(\cos\theta-\sin\theta, 2\sin\theta+2\cos\theta)\cdot\left(-\sin\theta, \frac{1}{2}\cos\theta\right)d\theta \\
&= \int_0^{2\pi} -\sin\theta\cos\theta+\sin^2\theta+\sin\theta\cos\theta+\cos^2\theta d\theta \\
&= \int_0^{2\pi} 1 d\theta = 2\pi
\end{aligned}$$

(b) No, since $\int_{C_1}\mathbf{F}\cdot d\mathbf{r}\neq 0$

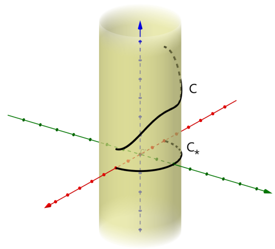
(c) Since $\mathbf{F}$ is conservative on every region which does not conatin $(0,0)$. Denote D is the region which is bounded by C_1 and C_2 does not contain $(0,0)$.

By Green's Theorem, we have $0 = \iint_D \dfrac{\partial Q}{\partial x} - \dfrac{\partial P}{\partial y} dA = \int_{C_1}\mathbf{F}\cdot d\mathbf{r} - \int_{C_2}\mathbf{F}\cdot d\mathbf{r}$

$$\Rightarrow \int_{C_2}\mathbf{F}\cdot d\mathbf{r} = \int_{C_1}\mathbf{F}\cdot d\mathbf{r} = 2\pi$$

where $(P,Q)=\mathbf{F}$.

7. (23%) Let S: $\{x^2+y^2=4,\ z\geq 0\}$ with unit normal $\mathbf{n}$ pointing outwards. Through S, a vector field $\mathbf{F}=y\,\mathbf{i}-x\,\mathbf{j}+x^2\,\mathbf{k}$ is also given.



(a) (5%) Write out $\mathbf{n}$ and $\operatorname{curl}\mathbf{F}$.

(b) (8%) As a surface integral, find $\iint_{S^*}(\operatorname{curl}\mathbf{F})\cdot\mathbf{n}\,dS$ where S^* is the part of S lying exactly beneath the parametric curve C depicted by $(x,\,y,\,z)=\mathbf{r}(t)=(2\cos(t),\,2\sin(t),\,3-2\cos^3(t)),\ 0\leq t\leq\pi$.

(c) (6%) Find $\int_C \mathbf{F}\cdot d\mathbf{r}$ and $\int_{C_*}\mathbf{F}\cdot d\mathbf{r}^*$, where C_* denotes the curve $\mathbf{r}^*(t)=(2\cos(t),\,2\sin(t),\,0),\ 0\leq t\leq\pi$.

(d) (4%) In addition to the two line integrals shown in (c), find the other two line integrals so that they collectively confirm the Stokes theorem.

Solution:

(a) $\mathbf{n}(x,\,y,\,z)=\dfrac{1}{2}(x,\,y,\,0)$ (2pts), $\nabla\times\mathbf{F}=-2x\,\mathbf{j}-2\,\mathbf{k}$ (3pts)

(b) Parametrize S^* by

$\mathbf{r}(t,z)=(2\cos t,2\sin t,z)$ where $0\leq t\leq\pi,\ 0\leq z\leq 3-2\cos^3(t)$ $\mathbf{r}_t\times\mathbf{r}_z=(2\cos t,2\sin t,0)$ (3pts)

$$\begin{aligned}\iint_{S^*}(\nabla\times\mathbf{F})\cdot\mathbf{n}\,dS &= \int_0^\pi\int_0^{3-2\cos^3(t)}(0,-2x,-2)\cdot(x,y,0)dzdt \text{ (2pts)}\\ &= -8\int_0^\pi \cos(t)\sin(t)(3-2\cos^3(t))dt\\ &= -8\int_{-1}^1 w(3-2w^3)dw = 32/5 \text{ (3pts)}\end{aligned}$$

(c)
$$\begin{aligned}\int_C \mathbf{F}\cdot d\mathbf{r} &= \int_0^\pi (y,-x,x^2)\cdot(-2\sin(t),2\cos(t),6\sin(t)\cos^2(t))dt \text{ (2pts)}\\ &= \int_0^\pi (y,-x,x^2)\cdot(-y,x,\tfrac{3}{4}x^2y)dt\\ &= \int_0^\pi(-4+24\cos^4(t)\sin(t))dt = 48/5-4\pi \text{ (2pts)}\end{aligned}$$
$$\begin{aligned}\int_{C_*}\mathbf{F}\cdot d\mathbf{r}_* &= \int_0^\pi (y,-x,x^2)\cdot(-2\sin(t),2\cos(t),0)dt\\ &= \int_0^\pi (y,-x,x^2)\cdot(-y,x,0)dt = -4\pi \text{ (2pts)}\end{aligned}$$

(d) Let C_1: $\mathbf{r}_1(t)=(2,\,0,\,t)$, C_2: $\mathbf{r}_2(t)=(-2,\,0,\,5t)$, $0\leq t\leq 1$.

We have $\int_{C_1}\mathbf{F}\cdot d\mathbf{r}_1=\int_0^1(0,\,-2,\,4)\cdot(0,\,0,\,1)dt=4$ (1pt)

$\int_{C_2}\mathbf{F}\cdot d\mathbf{r}_2=\int_0^1(0,\,2,\,4)\cdot(0,\,0,\,5)dt=20.$ (1pt)

So, suitable arrangement of these four integrals implies

$$\begin{aligned}\iint_{S^*}(\nabla\times\mathbf{F})\cdot\mathbf{n}\,dS &= \int_{C_*}\mathbf{F}\cdot d\mathbf{r}_* + \int_{C_2}\mathbf{F}\cdot d\mathbf{r}_2 - \int_C \mathbf{F}\cdot d\mathbf{r} - \int_{C_1}\mathbf{F}\cdot d\mathbf{r}\\ &= (-4\pi)+(20)-(48/5-4\pi)-(4) = 32/5 \text{ (2pts)}\end{aligned}$$

8. (14%) Let $\mathbf{F}(x,y,z) = (xy + axz)\,\mathbf{i} + (by^2 + \sqrt{2}yz)\,\mathbf{j} + 4yz\,\mathbf{k}$ where a, b are constants such that $\operatorname{div}\mathbf{F} = 0$.

(a) (3%) Find constants a, b.

(b) (3%) Show that if S_1 and S_2 are smooth oriented surface with same oriented boundary curve C, then $\iint_{S_1} \mathbf{F}\cdot d\mathbf{S} = \iint_{S_2} \mathbf{F}\cdot d\mathbf{S}$.

(c) (8%) Let S be the part of the sphere $x^2+y^2+z^2 = 1$ above the plane $z = 2\sqrt{2}y$ whose unit normal vectors point out of the unit ball. Find $\iint_S \mathbf{F}\cdot d\mathbf{S}$. (Hint : you may use the result from part (b)).

Solution:

(a) $\operatorname{div}\vec{F} = y + az + 2by + \sqrt{2}z + 4y = (5+2b)y + (a+\sqrt{2}z) = 0 \Rightarrow b = -\dfrac{5}{2}, a = -\sqrt{2}.$

(b) Let $-S_2$ be S_2 with opposite oreintantion. Then $S_1 \cup (-S_2)$ is a closed surface enclosing a solid E. Hence $\iint_{S_1\cup(-S_2)} \vec{F}\cdot d\vec{S} = \iiint_E \operatorname{div}\vec{F}dV = 0$ by the divergence theorem.
Thus,

$$\iint_{S_1\cup(-S_2)} \vec{F}\cdot d\vec{S} = \iint_{S_1} \vec{F}\cdot d\vec{S} + \iint_{-S_2} \vec{F}\cdot d\vec{S} = \iint_{S_1} \vec{F}\cdot d\vec{S} - \iint_{S_2} \vec{F}\cdot d\vec{S} = 0$$
$$\Rightarrow \iint_{S_1} \vec{F}\cdot d\vec{S} = \iint_{S_2} \vec{F}\cdot d\vec{S}.$$

(c) Let S_1 be the disc on the plane $z = 2\sqrt{2}y$ within the ball $x^2+y^2+z^2 \le 1$ with upward orientation.
Then S_1 and S_2 have same oriented boundary curve C. Hence

$$\iint_S \vec{F}\cdot d\vec{S} = \iint_{S_1} \vec{F}\cdot d\vec{S}$$

Now we parametrize S with $\vec{r}(x,y) = (x, y, 2\sqrt{2})$ where $x^2 + y^2 + (2\sqrt{2})^2y^2 \le 1$ i.e $x^2 + 9y^2 \le 1$
Let $D = \{(x,y) | x^2 + 9y^2 \le 1\}$. $\vec{r}_x \times \vec{r}_y = (0, -2\sqrt{2}, 1)$ is upward. Thus,

$$\begin{aligned}
\iint_{S_2} \vec{F}\cdot d\vec{S} &= \iint_D \vec{F}(\vec{r}(x,y))\cdot(\vec{r}_x\times\vec{r}_y)dxdy \\
&= \iint_D \left(-\frac{5}{2}y^2 + \sqrt{2}y\cdot 2\sqrt{2}y\right)(-2\sqrt{2}) + (4y\cdot 2\sqrt{2}y)\cdot 1dxdy \\
&= \iint_D 5\sqrt{2}y^2dxdy = 5\sqrt{2}\int_0^{2\pi}\int_0^1 \frac{1}{9}r^2\sin^2\theta rdrd\theta \\
&\qquad\qquad \begin{cases} x = r\cos\theta \\ y = \dfrac{1}{3}r\sin\theta \end{cases} \\
&= \frac{5\sqrt{2}}{36}\pi
\end{aligned}$$