微乙小考二 (2019/10/17)

1. 求以下的極限。

   (a) (3 pts) $\lim_{r\to 0}(1+23r)^{\frac{1}{r}}$.

   (b) (3 pts) $\lim_{h\to\infty}(h\cos(1/h)-h)$.

   Solution.

   (a) $\lim_{r\to 0}(1+23r)^{\frac{1}{r}} = \lim_{r\to 0}((1+23r)^{\frac{1}{23r}})^{23} = (\lim_{r\to 0}(1+23r)^{\frac{1}{23r}})^{23} = e^{23}$

   (b) $\lim_{h\to\infty}(h\cos(1/h)-h) = \lim_{h\to\infty}\dfrac{\cos(1/h)-1}{1/h} = \lim_{1/h\to 0}\dfrac{\cos(1/h)-1}{1/h} = \lim_{x\to 0}\dfrac{\cos(x)-1}{x} = \lim_{x\to 0}\dfrac{(\cos(x)-1)(\cos(x)+1)}{x(\cos(x)+1)}$
   $= \lim_{x\to 0}\dfrac{\cos^2(x)-1}{x(\cos(x)+1)} = \lim_{x\to 0}\dfrac{-\sin^2 x}{x(\cos(x)+1)} = \lim_{x\to 0}(-1)\cdot\dfrac{\sin x}{x}\cdot\sin x\cdot\dfrac{1}{\cos(x)+1} = (-1)\cdot 1\cdot 0\cdot\dfrac{1}{2} = 0$

2. (a) (3 pts) 令 $f(x)=\sin(\cos(2x^3))$, 求 $f'(x)$.

   (b) (3 pts) 令 $g(x)=e^{\tan^{-1}x}$, 求 $g'(x)$.

   (c) (3 pts) 令 $h(x)=x^{x^6}$, 求 $h'(1)$.

   Solution.

   (a) *let* $g(x)=\sin x, h(x)=\cos x, m(x)=2x^3$, *then* $f(x)=g(h(m(x)))$

   by chain rule,

   $f'(x)=g'(h(m(x)))\cdot h'(m(x))\cdot m'(x)$

   $\Rightarrow f'(x)=\cos(\cos(2x^3))(-\sin(2x^3))6x^2$

   (b) *let* $f(x)=e^x, h(x)=\tan^{-1}x$, *then* $g(x)=f(h(x))$

   by chain rule,

   $g'(x)=f'(h(x))\cdot h'(x)=\dfrac{e^{\tan^{-1}x}}{1+x^2}$

   (c) *let* $y=x^{x^6}\Rightarrow \ln y = x^6\ln x$

   $\Rightarrow \frac{y'}{y}=(6x^5\ln x + x^6\cdot\frac{1}{x})$

   $\Rightarrow y'=(6x^5\ln x + x^5)x^{x^6}$

   $\Rightarrow h'(1)=1$

3. (5 pts) 求曲線 $x^3-6yx^2+y^2x+2y^3=1$ 在點 $(1,0)$ 的切線方程式。

   Solution. $x^3-6yx^2+y^2x+2y^3=1$

   $\Rightarrow \frac{d}{dx}(x^3-6yx^2+y^2x+2y^3)=\frac{d}{dx}(1)$

   $\Rightarrow \frac{d}{dx}x^3-\frac{d}{dx}6yx^2+\frac{d}{dx}y^2x+\frac{d}{dx}2y^3=0$

   $\Rightarrow 3x^2-6(y'x^2+y\cdot 2x)+(2yy'x+y^2)+6y^2y'=0$

   $\Rightarrow y'=\dfrac{-3x^2+12xy-y^2}{-6x^2+2xy+6y^2}$

   plug in $(1,0)$

   $\Rightarrow y'=\frac{1}{2}$

   so the tangent line is $y=\frac{1}{2}(x-1)$