# 1081微乙01-04班期末考解答和評分標準

1. (10%) 令 $F(x) = \int_0^{x^2} \cos(t^2+t)\,\mathrm{d}t$.

   (a) (6%) 求 $F'(x)$.

   (b) (4%) 求極限 $\lim_{x\to 0} \frac{F(x)}{x^2}$.

**Solution:**

(a) Let $G(x) = \int_0^x \cos(t^2+t)dt$ and $h(x) = x^2$. Then $F(x) = G(h(x))$. By Fundamental Theorem of Calculus, one has $G'(x) = \cos(x^2+x)$. (2%) By Chain Rule, we have

$$F'(x) = G'(x)(h(x)) \cdot h'(x) = \cos(x^4+x^2)\cdot 2x. \quad (2\%,\ 2\%)$$

評分原則:
(i) 直接寫出正確答案者給滿分。
(ii) 知道要用微積分基本定理並列出相關函數者得兩分。
(iii) 沒使用連鎖律或使用錯誤者，依上面配分給分。

(b) By using L'Hospital rule (for the $\frac{0}{0}$ form), (1%) we have

$$\lim_{x\to 0}\frac{F(x)}{x^2} = \lim_{x\to 0}\frac{F'(x)}{2x} \quad (2\%)$$
$$= \lim_{x\to 0}\cos(x^4+x^2) = 1.\ (1\%)$$

2. (14%) 計算下列不定積分。

(a) (7%) $\int \sin\sqrt{2x+1}\,\mathrm{d}x$ (b) (7%) $\int \tan^{-1}\left(\frac{2}{x}\right)\mathrm{d}x$

**Solution:**

(a)

$$\left.\begin{aligned}&\text{Let } u=\sqrt{2x+1},\ du=\frac{1}{\sqrt{2x+1}}dx \text{ or } dx=udu\\&\int \sin\sqrt{2x+1}dx=\int(\sin u)\cdot udu\end{aligned}\right\}\ \text{3 pts for corect substitution}$$

$$=\int u\cdot d(-\cos u)=u\cdot(-\cos u)-\int(-\cos u)du \quad \text{(2 pts for integration by parts.)}$$

$$\underbrace{=-u\cos u+\sin u+C}_{\text{1 pt for }\int\cos u du=\sin u}=\underbrace{-\sqrt{2x+1}\cos(\sqrt{2x+1})+\sin(\sqrt{2x+1})+C}_{\text{1 pt}}$$

(b) $$\int \tan^{-1}\left(\frac{2}{x}\right)dx=x\cdot\tan^{-1}\left(\frac{2}{x}\right)-\int x\cdot d\left(\tan^{-1}\left(\frac{2}{x}\right)\right) \quad \text{(2 pts)}$$

$$\left.\begin{aligned}&=x\cdot\tan^{-1}\left(\frac{2}{x}\right)-\int x\cdot\frac{1}{1+\left(\frac{2}{x}\right)^2}\left(-\frac{2}{x^2}\right)dx\\&=x\cdot\tan^{-1}\left(\frac{2}{x}\right)+\int\frac{2x}{x^2+4}dx\end{aligned}\right\}\ \text{2 pts for }\frac{d}{dx}\left(\tan^{-1}\left(\frac{2}{x}\right)\right)$$

$$=x\cdot\tan^{-1}\left(\frac{2}{x}\right)+\ln(x^2+4)+C \quad \left(\text{2 pts for }\int\frac{2x}{x^2+4}dx\text{, 1 pt for final answer.}\right)$$

3. (18%) 計算下列不定積分。

(a) (9%) $\displaystyle\int \frac{1}{x^2\sqrt{1+x^2}}\,\mathrm{d}x$ (b) (9%) $\displaystyle\int \frac{x^2+4x-1}{x^4-1}\,\mathrm{d}x$

**Solution:**

(a)

$$\int \frac{1}{x^2\sqrt{1+x^2}}dx \qquad (\text{Let } x=\tan\theta,\ dx=\sec^2\theta d\theta)$$

$$\begin{aligned}
&= \int \frac{\sec^2\theta}{\tan^2\theta\sec\theta}d\theta \qquad (3\%)\\
&= \int \frac{\cos\theta}{\sin^2\theta}d\theta\\
&= \int \frac{1}{\sin^2\theta}d(\sin\theta) \quad (3\%)\\
&= -\frac{1}{\sin\theta}+C \quad (1\%)\\
&= -\frac{\sqrt{1+x^2}}{x^2}+C \quad (2\%)
\end{aligned}$$

(b) Write

$$\frac{x^2+4x-1}{x^4-1} = \frac{a}{x+1}+\frac{b}{x-1}+\frac{cx+d}{x^2+1}. \quad (1\%)$$

Then one has

$$x^2+4x-1 = a(x-1)(x^2+1)+b(x+1)(x^2+1)+(cx+d)(x^2-1).$$

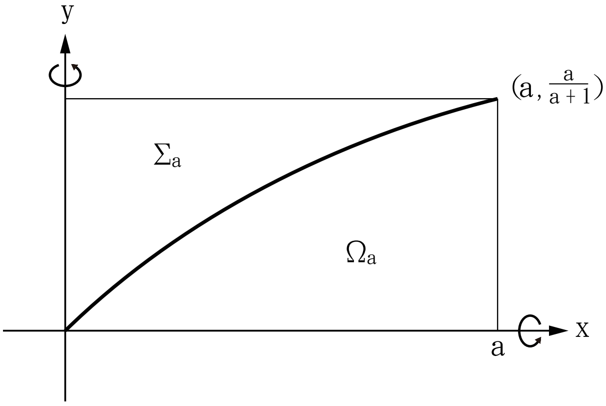
Put $x=1$, $b=1$. Put $x=-1$, $a=1$. Compare the coefficients of $x^3$ and the constants on the both sides, one has $c=-1$ and $d=1$. (4%; 每個變數各一分) So we have

$$\frac{x^2+4x-1}{x^4-1} = \frac{1}{x+1}+\frac{1}{x-1}+\frac{-2x+1}{x^2+1}.$$

Than

$$\begin{aligned}
\int \frac{x^2+4x-1}{x^4-1}dx &= \int\left(\frac{1}{x+1}+\frac{1}{x-1}+\frac{-2x}{x^2+1}+\frac{1}{x^2+1}\right)dx\\
&= \ln|x+1|+\ln|x-1|-\ln(x^2+1)+\tan^{-1}x. \qquad (4\%;\ \text{每個積分各一分})
\end{aligned}$$

4. (18%) 已知$a > 0$. 令 $\Omega_a$: $y = \dfrac{x}{x+1}$, $x$-軸, $x = a$ 圍成的區域; $\Sigma_a$: $y = \dfrac{x}{x+1}$, $y$-軸, $y = \dfrac{a}{a+1}$ 圍成的區域.($\Omega_a$和$\Sigma_a$合成一長方形如圖). 令 $U(a)$ 為 $\Omega_a$ 繞 $x$-軸旋轉之旋轉體體積; $V(a)$ 為 $\Sigma_a$ 繞 $y$-軸旋轉之旋轉體體積。




(a) (8%) 求 $U(a)$.

(b) (8%) 求 $V(a)$.

(c) (2%) 求 $U(a) - V(a)$.

**Solution:**

([8%]+[8%])會圓盤法或殼形法公式2%; 積出不定積分部分4%; 算值正確2%。

$$\begin{aligned}
U(a) &= \int_0^a \pi\left(\frac{x}{x+1}\right)^2 dx[2\%] \\
&= \pi\int_0^a \left(1-\frac{1}{x+1}\right)^2 dx \\
&= \pi\int_0^a 1-\frac{2}{x+1}+\frac{1}{(x+1)^2}\,dx[6\%] \\
&= \pi\left(a-2\ln(a+1)+(-\frac{1}{a+1}+1)\right) \\
&= \pi\left((a+1-\frac{1}{a+1})-2\ln(a+1)\right)[8\%] \\
V(a) &= \int_0^a 2\pi x(\frac{a}{a+1}-\frac{x}{x+1})\,dx[2\%] \\
&= 2\pi\left(\int_0^a \frac{a}{a+1}x\,dx-\int_0^a \frac{x^2}{x+1}\,dx\right) \\
&= 2\pi\left(\int_0^a \frac{a}{a+1}x\,dx-\int_0^a x-1+\frac{1}{x+1}\,dx\right)[6\%] \\
&= \pi\left((a+1-\frac{1}{a+1})-2\ln(a+1)\right)[8\%]
\end{aligned}$$

[2%]所以對任何$a$, $U(a) - V(a) = 0$.

5. (12%) 求下列極限。

(a) (6%) $\lim_{x\to 0}\frac{\cos x^2-1}{\sin x^4}$　　(b) (6%) $\lim_{x\to 1}\left(\frac{2}{x^2-1}-\frac{1}{\ln x}\right)$

**Solution:**

(a) 使用羅必達時要說明type $\frac{0}{0}$, $\frac{\infty}{\infty}$, 沒寫 type 扣一分, 因 type 沒寫單小題最多扣兩分。

Method 1.

$$\begin{aligned}
&\lim_{x\to 0}\frac{\cos x^2-1}{\sin x^4}\\
&=\lim_{x\to 0}\left(\frac{\cos x^2-1}{\sin x^4}\frac{\sin x^4}{x^4}\right) \quad 2 \text{ point}\\
&=\lim_{x\to 0}\frac{\cos x^2-1}{x^4} \quad 1 \text{ point}\\
&=\lim_{x\to 0}\frac{-2x\sin x^2}{4x^3} \quad 1 \text{ point}\\
&=-\frac{1}{2}\lim_{x\to 0}\frac{\sin x^2}{x^2} \quad 1 \text{ point}\\
&=-\frac{1}{2} \quad 1 \text{ point.}
\end{aligned}$$

Method 2.

$$\begin{aligned}
&\lim_{x\to 0}\frac{\cos x^2-1}{\sin x^4}\\
&=\lim_{x\to 0}\frac{-2x\sin x^2}{4x^3\cos x^4} \quad 2 \text{ point}\\
&=-\frac{1}{2}\lim_{x\to 0}\left(\frac{\sin x^2}{x^2}\frac{1}{\cos x^4}\right) \quad 2 \text{ point}\\
&=-\frac{1}{2}\lim_{x\to 0}\left(\frac{\sin x^2}{x^2}\right)\lim_{x\to 0}\left(\frac{1}{\cos x^4}\right) \quad 1 \text{ point}\\
&=-\frac{1}{2} \quad 1 \text{ point.}
\end{aligned}$$

Method 3.

$$\begin{aligned}
\lim_{x\to 0}\frac{\cos x^2-1}{\sin x^4}&=\lim_{x\to 0}\frac{-\frac{x^4}{2!}+\frac{x^8}{4!}-\cdots}{x^4-\frac{x^{12}}{3!}+\cdots} \quad 4 \text{ point}\\
&=\lim_{x\to 0}\frac{-\frac{1}{2!}+\frac{x^4}{4!}-\cdots}{1-\frac{x^8}{3!}+\cdots} \quad 1 \text{ point}\\
&=-\frac{1}{2} \quad 1 \text{ point.}
\end{aligned}$$

(b) Method 1

$$\begin{aligned}\lim_{x\to 1}\left(\frac{2}{x^2-1}-\frac{1}{\ln x}\right) &= \lim_{x\to 1}\frac{2\ln x-(x^2-1)}{(x^2-1)\ln x} \quad \text{2 point.}\\ &= \lim_{x\to 1}\frac{2/x-2x}{2x\ln x+(x^2-1)/x} \quad \text{2 point.}\\ &= \lim_{x\to 1}\frac{-2/x^2-2}{2\ln x+2+1+1/x^2} \quad \text{1 point.}\\ &= -1 \quad \text{1 point.}\end{aligned}$$

此處必須說明 (各一分)

$$\lim_{x\to 1}(x^2-1)\ln x = 0 = \lim_{x\to 1} x\ln x = 0.$$

才能直接在上面計算用羅必達.

Method 2

$$\begin{aligned}\lim_{x\to 1}\left(\frac{2}{x^2-1}-\frac{1}{\ln x}\right) &= \lim_{x\to 1}\frac{\ln x^2-(x^2-1)}{(x^2-1)\ln x} \quad \text{1 point}\\ &= \lim_{x\to 1}\frac{-\frac{1}{2}(x^2-1)^2+\frac{1}{3}(x^2-1)^3-\cdots}{(x^2-1)\ln x} \quad \text{2 point}\\ &\lim_{x\to 1}\frac{(x^2-1)}{\ln x}\cdot\lim_{x\to 1}(-\frac{1}{2}+\frac{1}{3}(x^2-1)^2)-\cdots \quad \text{1 point}\\ &= \lim_{x\to 1}\frac{2x}{1/x}\cdot\frac{-1}{2} \quad \text{1 point}\\ &= -1 \quad \text{1 point.}\end{aligned}$$

6. (a) (4%) 寫下 $\dfrac{-1}{\sqrt{1-x^2}}$ 對 $x=0$ 的泰勒展式。

(b) (6%) 求 $\cos^{-1} x$ 對 $x=0$ 的泰勒展式。

(c) (4%) 求 $\cos^{-1}(x^2)$ 在 $x=0$ 的 10 次泰勒多項式。

**Solution:**

(a)

$$\frac{-1}{\sqrt{1-x^2}} = -(1-x^2)^{-1/2}$$
$$= -\sum_{k=0}^{\infty}(-1)^k C_k^{-1/2} x^{2k} \quad \text{4 point.}$$

**正負號不對扣一分**.

(b) Take integral on both sides above (2 point) , we have

$$\cos^{-1} x = c - \sum_{k=0}^{\infty} \frac{1}{2k+1}(-1)^k C_k^{-1/2} x^{2k+1} \quad \text{3 point.}$$

Let $x=0$, we have $c = \pi/2$ 1 point.

(c) From (b), we have

$$\cos^{-1} x^2 = \frac{\pi}{2} - \sum_{k=0}^{\infty}(-1)^k C_k^{-1/2} \frac{x^{4k+2}}{2k+1}.$$

$$P_{10}(x) = \frac{\pi}{2} - \sum_{k=0}^{2}(-1)^k C_k^{-1/2} \frac{x^{4k+2}}{2k+1}$$
$$= \frac{\pi}{2} - x^2 - \frac{1}{6}x^6 - \frac{3}{40}x^{10} \quad \text{1 point for each term.}$$

7. (14%) 若 $f(x) = \frac{1}{2}x^2 + \frac{1}{3\cdot 2}x^3 + \cdots + \frac{1}{n(n-1)}x^n + \cdots$, 在 $|x| < 1$.

(a) (2%) 求 $f^{(10)}(0)$

(b) (6%) 求 $f'(x)$ 和 $f''(x)$ 在 $x = 0$ 的泰勒展式，並認出它們是什麼基本函數。

(c) (6%) 把 $f(x)$ 表示成基本函數。

**Solution:**

(a) The coefficient of the term $x^{10}$ in the Taylor series of $f(x)$ at $x = 0$ is $\frac{f^{(10)}(0)}{10!} = \frac{1}{10\cdot 9} \Rightarrow$ $f^{(10)}(0) = \frac{10!}{10\cdot 9} = 8!$ (2 pts)

(b) The Taylor series of $f'(x)$ at $x = 0$ is

$x + \frac{1}{2}x^2 + \cdots + \frac{1}{n}x^n + \cdots$ (2 pts)

$x + \frac{1}{2}x^2 + \cdots + \frac{1}{n}x^n + \cdots = -\ln(1-x)$ (1 pt)

The Taylor series of $f''(x)$ at $x = 0$ is

$1 + x + x^2 + \cdots + x^n + \cdots$ (1 pt)

$1 + x + x^2 + \cdots + x^n + \cdots = \frac{1}{1-x}$ (2 pts)

(c) $f(x) = \int_0^x f'(t)dt + f(0)$ (1 pt)

Since $f(0) = 0, wehave$
$f(x) = \int_0^x f'(t)dt$ } (1 pt for $f(0) = 0$)

$= \int_0^x -\ln(1-t)dt = -\left[t\ln(1-t)\Big|_{t=0}^{t=x} - \int_0^x t\frac{-1}{1-t}dt\right]$ (2 pts for integration by parts)

$= -x\ln(1-x) + \int_0^x \frac{t}{t-1}dt$
$= -x\ln(1-x) + \int_0^x 1 + \frac{1}{t-1}dt$
$= -x\ln(1-x) + x + \ln|x-1|$ } (2 pts)