1. (14 pts) Find the limit, if it exists.

(a) (4 pts) $\lim_{x\to 0} \dfrac{\sin(x^2)}{1-\cos x}$.

(b) (5 pts) $\lim_{x\to 0} \dfrac{|\tan x|}{1-\sqrt{1+2x}}$.

(c) (5 pts) $\lim_{x\to\infty} \left(1+\dfrac{2}{x}\right)^{[x]}$.

$\left(\text{Hint: } [x] \text{ is the greatest integer function, } x-1 < [x] \le x \text{ and } \lim_{x\to\infty}\left(1+\dfrac{1}{x}\right)^x = e.\right)$

Solution:

(a) (Method 1)

$$\lim_{x\to 0}\frac{\sin(x^2)}{1-\cos x} \overset{(\frac{0}{0})}{=} \lim_{x\to 0}\frac{2x\cdot\cos(x^2)}{\sin x} \quad \text{(2 pts)}$$
$$\overset{(\frac{0}{0})}{=} \lim_{x\to 0}\frac{2\cos(x^2)-4x^2\cdot\sin(x^2)}{\cos x} \quad \text{(1 pt)}$$
$$= \frac{2}{1} = 2. \quad \text{(1 pt)}$$

(Method 2)

$$\lim_{x\to 0}\frac{\sin(x^2)}{1-\cos x} \overset{(\frac{0}{0})}{=} \lim_{x\to 0}\frac{2x\cos(x^2)}{\sin x} \quad \text{(2 pts)}$$
$$= (\lim_{x\to 0}\frac{x}{\sin x})\cdot(\lim_{x\to 0} 2\cos(x^2)) \quad \text{(1 pt)}$$
$$= 1\cdot 2 = 2. \quad \text{(1 pt)}$$

(Method 3)

$$\lim_{x\to 0}\frac{\sin(x^2)}{1-\cos x} = \lim_{x\to 0}\frac{\sin(x^2)\cdot(1+\cos x)}{(1-\cos x)(1+\cos x)} = \lim_{x\to 0}\frac{\sin(x^2)\cdot(1+\cos x)}{\sin^2 x} \quad \text{(2 pts)}$$
$$= \lim_{x\to 0}\frac{\sin(x^2)}{x^2}\cdot\frac{x^2}{\sin^2 x}\cdot(1+\cos x) \quad \text{(1 pt)}$$
$$= (\lim_{x\to 0}\frac{\sin(x^2)}{x^2})\cdot(\lim_{x\to 0}\frac{x^2}{\sin^2 x})\cdot(\lim_{x\to 0} 1+\cos x) = 1\cdot\cdot 1^2\cdot 2 = 2. \quad \text{(1 pt)}$$

(Method 4)

$$\lim_{x\to 0}\frac{\sin(x^2)}{1-\cos x} = \lim_{x\to 0}\frac{\sin(x^2)\cdot(1+\cos x)}{(1-\cos x)(1+\cos x)} = \lim_{x\to 0}\frac{\sin(x^2)\cdot(1+\cos x)}{\sin^2 x} \quad \text{(2 pts)}$$
$$= \lim_{x\to 0}\frac{\sin(x^2)}{\sin^2 x}\cdot(1+\cos x) = (\lim_{x\to 0}\frac{\sin(x^2)}{x^2})\cdot(\lim_{x\to 0} 1+\cos x) \quad \text{(1 pt)}$$
$$\overset{(\frac{0}{0})}{=} (\lim_{x\to 0}\frac{2x\cos(x^2)}{2x})\cdot 2 = (\lim_{x\to 0}\frac{\cos(x^2)}{1})\cdot 2 = 1\cdot 2 = 2. \quad \text{(1 pt)}$$

P.S. If you use L'Hôspital rule without check the condition $\frac{0}{0}$ or $\frac{\infty}{\infty}$, then you at most get 3 points in this question.

(b) (Method 1)

We should find both limits $\lim\limits_{x\to 0^+}\dfrac{|\tan x|}{1-\sqrt{1+2x}}$ and $\lim\limits_{x\to 0^-}\dfrac{|\tan x|}{1-\sqrt{1+2x}}$. Since

$$\lim_{x\to 0^+}\frac{|\tan x|}{1-\sqrt{1+2x}} = \lim_{x\to 0^+}\frac{\tan x}{1-\sqrt{1+2x}} \overset{(\frac{0}{0})}{=} \lim_{x\to 0^+}\frac{\sec^2 x}{\frac{-1}{2}\cdot(1+2x)^{-1/2}\cdot 2} \quad \text{(1 pt)}$$
$$= \lim_{x\to 0^-} -\sqrt{1+2x}\cdot\sec^2 x = -1, \quad \text{(1 pt)}$$

and

$$\lim_{x\to 0^-}\frac{|\tan x|}{1-\sqrt{1+2x}} = \lim_{x\to 0^-}\frac{-\tan x}{1-\sqrt{1+2x}} \overset{(\frac{0}{0})}{=} \lim_{x\to 0^-}\frac{-\sec^2 x}{\frac{-1}{2}\cdot(1+2x)^{-1/2}\cdot 2} \quad \text{(1 pt)}$$
$$= \lim_{x\to 0^-} \sqrt{1+2x}\cdot\sec^2 x = 1, \quad \text{(1 pt)}$$

the limit $\lim\limits_{x\to 0}\dfrac{|\tan x|}{1-\sqrt{1+2x}}$ does not exist. (1 pt)

(Method 2)

We should find both limits $\lim\limits_{x\to 0^+}\dfrac{|\tan x|}{1-\sqrt{1+2x}}$ and $\lim\limits_{x\to 0^-}\dfrac{|\tan x|}{1-\sqrt{1+2x}}$. Since

$$\lim_{x\to 0^+}\frac{|\tan x|}{1-\sqrt{1+2x}} = \lim_{x\to 0^+}\frac{(\tan x)\cdot(1+\sqrt{1+2x})}{(1-\sqrt{1+2x})(1+\sqrt{1+2x})} = \lim_{x\to 0^+}\frac{(\tan x)\cdot(1+\sqrt{1+2x})}{-2x} \quad \text{(1 pt)}$$
$$= \lim_{x\to 0^+}\frac{\sin x}{x}\cdot\frac{1+\sqrt{1+2x}}{-2\cos x} = (\lim_{x\to 0^+}\frac{\sin x}{x})\cdot(\lim_{x\to 0^+}\frac{1+\sqrt{1+2x}}{-2\cos x}) = 1\cdot\frac{2}{-2} = -1, \quad \text{(1 pt)}$$

and

$$\lim_{x\to 0^-}\frac{|\tan x|}{1-\sqrt{1+2x}} = \lim_{x\to 0^-}\frac{(-\tan x)\cdot(1+\sqrt{1+2x})}{(1-\sqrt{1+2x})(1+\sqrt{1+2x})} = \lim_{x\to 0^-}\frac{(-\tan x)\cdot(1+\sqrt{1+2x})}{-2x} \quad \text{(1 pt)}$$
$$= \lim_{x\to 0^-}\frac{\sin x}{x}\cdot\frac{1+\sqrt{1+2x}}{2\cos x} = (\lim_{x\to 0^-}\frac{\sin x}{x})\cdot(\lim_{x\to 0^-}\frac{1+\sqrt{1+2x}}{2\cos x}) = 1\cdot\frac{2}{2} = 1, \quad \text{(1 pt)}$$

the limit $\lim\limits_{x\to 0}\dfrac{|\tan x|}{1-\sqrt{1+2x}}$ does not exist. (1 pt)

(Method 3)

We should find both limits $\lim\limits_{x\to 0^+}\dfrac{|\tan x|}{1-\sqrt{1+2x}}$ and $\lim\limits_{x\to 0^-}\dfrac{|\tan x|}{1-\sqrt{1+2x}}$. Since

$$\lim_{x\to 0^+}\frac{|\tan x|}{1-\sqrt{1+2x}} = \lim_{x\to 0^+}\frac{(\tan x)\cdot(1+\sqrt{1+2x})}{(1-\sqrt{1+2x})(1+\sqrt{1+2x})} = \lim_{x\to 0^+}\frac{(\tan x)\cdot(1+\sqrt{1+2x})}{-2x} \quad \text{(1 pt)}$$
$$= \lim_{x\to 0^+}\frac{\sin x}{x}\cdot\frac{1+\sqrt{1+2x}}{-2\cos x} \overset{(\frac{0}{0})}{=} (\lim_{x\to 0^+}\frac{\cos x}{1})\cdot(\lim_{x\to 0^-}\frac{1+\sqrt{1+2x}}{-2\cos x}) = 1\cdot\frac{2}{-2} = -1, \quad \text{(1 pt)}$$

and

$$\lim_{x\to 0^-}\frac{|\tan x|}{1-\sqrt{1+2x}} = \lim_{x\to 0^-}\frac{(-\tan x)\cdot(1+\sqrt{1+2x})}{(1-\sqrt{1+2x})(1+\sqrt{1+2x})} = \lim_{x\to 0^-}\frac{(-\tan x)\cdot(1+\sqrt{1+2x})}{-2x} \quad \text{(1 pt)}$$
$$= \lim_{x\to 0^-}\frac{\sin x}{x}\cdot\frac{1+\sqrt{1+2x}}{2\cos x} \overset{(\frac{0}{0})}{=} (\lim_{x\to 0^-}\frac{\cos x}{1})\cdot(\lim_{x\to 0^-}\frac{1+\sqrt{1+2x}}{2\cos x}) = 1\cdot\frac{2}{2} = 1, \quad \text{(1 pt)}$$

the limit $\lim\limits_{x\to 0}\dfrac{|\tan x|}{1-\sqrt{1+2x}}$ does not exist. (1 pt)

(c) (Method 1)

Since $x-1<[x]\le x$, we have $(1+\dfrac{2}{x})^{x-1}<(1+\dfrac{2}{x})^{[x]}\le(1+\dfrac{2}{x})^{x}$ for $x>0$. We need to find $\lim\limits_{x\to\infty}(1+\dfrac{2}{x})^{x-1}$ and $\lim\limits_{x\to\infty}(1+\dfrac{2}{x})^{x}$. The following are the computations of both limits:

$$\lim_{x\to\infty}(1+\frac{2}{x})^{x}=\lim_{x\to\infty}[(1+\frac{2}{x})^{\frac{x}{2}}]^2=[\lim_{x\to\infty}(1+\frac{2}{x})^{\frac{x}{2}}]^2=e^2. \qquad \text{(2 pts)}$$

And

$$\lim_{x\to\infty}(1+\frac{2}{x})^{x-1}=\lim_{x\to\infty}(1+\frac{2}{x})^{x}\cdot\lim_{x\to\infty}(1+\frac{2}{x})^{-1}=e^2\cdot 1=e^2. \qquad \text{(2 pts)}$$

Therefore, we get $e^2=\lim\limits_{x\to\infty}(1+\dfrac{2}{x})^{x-1}\le\lim\limits_{x\to\infty}(1+\dfrac{2}{x})^{[x]}\le\lim\limits_{x\to\infty}(1+\dfrac{2}{x})^{x}=e^2$. By squeeze theorem, we have $\lim\limits_{x\to\infty}(1+\dfrac{2}{x})^{[x]}=e^2$. (1 pt)

(Method 2)

Let $y=(1+\dfrac{2}{x})^{[x]}$. Then $\ln y=[x]\cdot\ln(1+\dfrac{2}{x})$. Since $x-1<[x]\le x$, we have $(x-1)\cdot\ln(1+\dfrac{2}{x})<[x]\cdot\ln(1+\dfrac{2}{x})\le x\cdot\ln(1+\dfrac{2}{x})$. We need to compute both limits $\lim\limits_{x\to\infty}(x-1)\cdot\ln(1+\dfrac{2}{x})$ and $\lim\limits_{x\to\infty}x\cdot\ln(1+\dfrac{2}{x})$.

$$\lim_{x\to\infty}(x-1)\cdot\ln(1+\frac{2}{x})=\lim_{x\to\infty}\frac{\ln(1+\frac{2}{x})}{(x-1)^{-1}}\overset{(\frac{0}{0})}{=}\lim_{x\to\infty}\frac{-2x^{-2}}{(-1)(x-1)^{-2}\cdot(1+\frac{2}{x})}$$
$$=\lim_{x\to\infty}\frac{2(x-1)^2}{x^2(1+\frac{2}{x})}=\lim_{x\to\infty}\frac{2(1-\frac{1}{x})^2}{1+\frac{2}{x}}=2. \qquad \text{(2 pts)}$$

On the other hand,

$$\lim_{x\to\infty}x\cdot\ln(1+\frac{2}{x})=\lim_{x\to\infty}\frac{\ln(1+\frac{2}{x})}{x^{-1}}\overset{(\frac{0}{0})}{=}\lim_{x\to\infty}\frac{-2x^{-2}}{(-1)x^{-2}\cdot(1+\frac{2}{x})}$$
$$=\lim_{x\to\infty}\frac{2}{(1+\frac{2}{x})}=2. \qquad \text{(2 pts)}$$

Therefore, we get $\lim\limits_{x\to\infty}(x-1)\cdot\ln(1+\dfrac{2}{x})\le\lim\limits_{x\to\infty}[x]\cdot\ln(1+\dfrac{2}{x})\le\lim\limits_{x\to\infty}x\cdot\ln(1+\dfrac{2}{x})$. By squeeze theorem, we have $\lim\limits_{x\to\infty}\ln y=\lim\limits_{x\to\infty}[x]\cdot\ln(1+\dfrac{2}{x})=2$ which implies that $\lim\limits_{x\to\infty}y=\lim\limits_{x\to\infty}(1+\dfrac{2}{x})^{[x]}=e^2$.
(1 pt)

(Method 3)

Since $x-1<[x]\le x$, we have $(1+\dfrac{2}{x})^{x-1}<(1+\dfrac{2}{x})^{[x]}\le(1+\dfrac{2}{x})^{x}$ for $x>0$. So we need to compute $\lim\limits_{x\to\infty}(1+\dfrac{2}{x})^{x-1}$ and $\lim\limits_{x\to\infty}(1+\dfrac{2}{x})^{x}$. Note that

$$\lim_{x\to\infty}(1+\frac{2}{x})^{x-1}=\lim_{x\to\infty}e^{(x-1)\cdot\ln(1+\frac{2}{x})}=e^{\lim\limits_{x\to\infty}(x-1)\cdot\ln(1+\frac{2}{x})}$$

and

$$\lim_{x\to\infty}(1+\frac{2}{x})^{x}=\lim_{x\to\infty}e^{x\cdot\ln(1+\frac{2}{x})}=e^{\lim\limits_{x\to\infty}x\cdot\ln(1+\frac{2}{x})}.$$

Hence it suffices to find both limits $\lim_{x\to\infty}(x-1)\cdot\ln(1+\frac{2}{x})$ and $\lim_{x\to\infty}x\cdot\ln(1+\frac{2}{x})$.

$$\lim_{x\to\infty}(x-1)\cdot\ln(1+\frac{2}{x}) = \lim_{x\to\infty}\frac{\ln(1+\frac{2}{x})}{(x-1)^{-1}} \overset{(\frac{0}{0})}{=} \lim_{x\to\infty}\frac{-2x^{-2}}{(-1)(x-1)^{-2}\cdot(1+\frac{2}{x})}$$
$$= \lim_{x\to\infty}\frac{2(x-1)^2}{x^2(1+\frac{2}{x})} = \lim_{x\to\infty}\frac{2(1-\frac{1}{x})^2}{1+\frac{2}{x}} = 2. \qquad \text{(2 pts)}$$

On the other hand,

$$\lim_{x\to\infty}x\cdot\ln(1+\frac{2}{x}) = \lim_{x\to\infty}\frac{\ln(1+\frac{2}{x})}{x^{-1}} \overset{(\frac{0}{0})}{=} \lim_{x\to\infty}\frac{-2x^{-2}}{(-1)x^{-2}\cdot(1+\frac{2}{x})}$$
$$= \lim_{x\to\infty}\frac{2}{(1+\frac{2}{x})} = 2. \qquad \text{(2 pts)}$$

Therefore, we get $e^2 = \lim_{x\to\infty}(1+\frac{2}{x})^{x-1} \leq \lim_{x\to\infty}(1+\frac{2}{x})^{[x]} \leq \lim_{x\to\infty}(1+\frac{2}{x})^{x} = e^2$. By squeeze theorem, we have $\lim_{x\to\infty}(1+\frac{2}{x})^{[x]} = e^2$. (1 pt)

2. (14 pts)

(a) (4 pts) $f(x) = e^{x^2-x}$. Find $f'(x)$ and $f''(x)$.

(b) (4 pts) $f(x) = \sin^{-1}(\sqrt{1-x^2})$. Find $f'(x)$.

(c) (6 pts) $f(x) = (\cos x)^{\log_2 x} + x^2 \cdot \sec x$. Find $f'(x)$.

Solution:

(a) $f(x) = e^{x^2-x}$. The first derivative**(2pts)** is given by

$$f'(x) = \frac{de^{x^2}}{d(x^2-x)} \cdot \frac{d(x^2-x)}{dx} = e^{x^2-x} \cdot (2x-1) = (2x-1)e^{x^2-x}.$$

The second derivative**(2pts)** is given by

$$\begin{aligned} f''(x) &= \frac{d(2x-1)e^{x^2-x}}{dx} = \frac{d(2x-1)}{dx}e^{x^2-x} + (2x-1)\frac{de^{x^2-x}}{dx} \\ &= 2e^{x^2-x} + (2x-1)^2 e^{x^2-x} = (4x^2-4x+3)e^{x^2-x}. \end{aligned}$$

Partial credits:

- If the calculation of the first derivative is wrong, there will be no point for the second derivative part. However, the student may still get **1pt** for showing some technique of differentiation.

(b) $f(x) = \sin^{-1}(\sqrt{1-x^2})$. The formula for differentiating $\sin^{-1}(x)$**(1pt)** is

$$\frac{d}{dx}\sin^{-1}(x) = \frac{1}{\sqrt{1-x^2}}.$$

Therefore, using chain rule, we have the derivative**(2pts)**

$$\begin{aligned} \frac{d}{dx}\sin^{-1}(\sqrt{1-x^2}) &= \frac{d\sin^{-1}(\sqrt{1-x^2})}{d\sqrt{1-x^2}} \cdot \frac{d\sqrt{1-x^2}}{d(1-x^2)} \cdot \frac{d(1-x^2)}{dx} \\ &= \frac{1}{\sqrt{1-(\sqrt{1-x^2})}} \cdot \frac{1}{2\sqrt{1-x^2}} \cdot (-2x) \\ &= \frac{-x}{\sqrt{x^2}\sqrt{1-x^2}} = \frac{-x}{|x|\sqrt{1-x^2}} = \frac{-\mathbf{sgn}(x)}{\sqrt{1-x^2}} \end{aligned}$$

Note that any answer appeared in the last line of the equation is acceptable.

Partial credits:

- If the student cancels x and $|x|$, **1pt** will be taken off.

(c) $f(x) = (\cos x)^{\log_2 x} + x^2 \cdot \sec x$. The derivative of the first half**(3pts)** is given by

$$\begin{aligned} \frac{d(\cos x)^{\log_2 x}}{dx} &= \frac{de^{\ln\cos x \log_2 x}}{dx} = \frac{de^{\ln\cos x \log_2 x}}{d(\ln\cos x \log_2 x)} \cdot \frac{d(\ln\cos x \log_2 x)}{dx} \\ &= e^{\ln\cos x \log_2 x} \cdot \left(\frac{-\sin x}{\cos x}\log_2 x + \ln(\cos x)\frac{1}{(\ln 2)x}\right) \\ &= (\cos x)^{\log_2 x}\left(-\tan x \log_2 x + \frac{\log_2(\cos x)}{x}\right). \end{aligned}$$

The derivative of the second half **(3pts)** is given by

$$\begin{aligned}\frac{d(x^2 \sec x)}{dx} &= \frac{dx^2}{dx} \sec x + x^2 \frac{d \sec x}{dx} \\ &= 2x \sec x + x^2 \sec x \tan x.\end{aligned}$$

Therefore, the complete answer is given by

$$f'(x) = (\cos x)^{\log_2 x}(-\tan x \log_2 x + \frac{\log_2(\cos x)}{x}) + 2x \sec x + x^2 \sec x \tan x.$$

3. (12 pts)

(a) Suppose that a function f has the property:

$$|f(x_1) - f(x_2)| \le |x_1 - x_2|^2 \text{ for any real number } x_1,\, x_2.$$

i. (5 pts) Show that f is differentiable everywhere.

ii. (2 pts) Determine f explicitly.

(b) (5 pts) Suppose now that another function g has the property:

$$g(3x) = 2(g(x) + x) \text{ for any real number } x, \text{ and } g \text{ is differentiable at } x = 0.$$

Find $g(0)$ and $g'(0)$.

Solution:

(a) i. (2%) Set $x_1 = x + h,\ x_2 = x \Rightarrow |f(x+h) - f(x)| \le |h|^2$ for any x

(1%) $\Rightarrow -h \le \dfrac{f(x+h) - f(x)}{h} \le h$

(1%) $\Rightarrow \lim\limits_{h\to 0}(-h) \le \lim\limits_{h\to 0} \dfrac{f(x+h) - f(x)}{h} \le \lim\limits_{h\to 0} h$

(1%) $\Rightarrow f'(x) = 0$ for any x, by squeezing.

ii. (2%) Since $f'(x) = 0$ for any x, $f(x) = C$ for all x where C is any constant, as an application of Mean Value Theorem.

(b) (1%) Set $x = 0 \Rightarrow g(0) = 2(g(0) + 0) \Rightarrow g(0) = 0$

(1%) g is differentiable at $x = 0 \Rightarrow$ chain rule is applicable

(2%) $\Rightarrow 3g'(0) = 2(g'(0) + 1)$

(1%) $\Rightarrow g'(0) = 2$

4. (12 pts) Let $f(x) = \tan^{-1} x + 2x$.

(a) (4 pts) Show that $f(x)$ is one to one. Therefore $f(x)$ has inverse function.

(b) (4 pts) Find $f^{-1}\left(\frac{\pi}{4}+2\right)$ and $\frac{d}{dx}f^{-1}\Big|_{\frac{\pi}{4}+2}$.

(c) (4 pts) Write down the linear approximation of $f^{-1}(x)$ at $x = \frac{\pi}{4}+2$. Use the linear approximation to estimate $f^{-1}\left(\frac{\pi}{4}+1.95\right)$.

Solution:

(a) We compute that $f'(x) = \frac{1}{1+x^2} + 2$. (1%)
Then we have $f'(x) \geq 2$ for all $x \in \mathbb{R}$. (1%)
Method 1.(Use the increasing property to obtain one to one):
Therefore, f is increasing. (1%)
This implies that $f(x)$ is one to one. (1%)
Method 2.(Use Rolle's theorem):
Assume there are two numbers $x_1 < x_2$ such that $f(x_1) = f(x_2)$. Since f is continuous and differentiable for all x, (1%)
by applying the Rolle's theorem, there is number c between x_1 and x_2 such that $f'(c) = 0$. It contradicts to $f'(x) \geq 2$ for all $x \in \mathbb{R}$. Therefore, $f(x)$ is one to one. (1%)
Method 3.(Use Mean Value Theorem):
For any $x_1 \neq x_2$, we want to show $f(x_1) \neq f(x_2)$. WLOG, we assume that $x_2 > x_1$. Since f is continuous and differentiable for all x(1%),
by applying the Mean Value Theorem, there is number c between x_1 and x_2 such that $f(x_2) - f(x_1) = f'(c)(x_2 - x_1)$. (1%)
Therefore, we have $f(x_2) - f(x_1) \geq 2(x_2 - x_1) > 0$ and $f(x)$ is one to one. (1%)
Method 4.:
Since $(\tan^{-1} x)' = \frac{1}{1+x^2} > 0$ and $(2x)' = 2$, we have $\tan^{-1} x$ and $2x$ are strictly increasing. (2%)
This implies that $\tan^{-1} x + 2x$ is strictly increasing. (1%)
Therefore, the function $f(x)$ is one to one. (1%)

(b) Since $f(1) = \frac{\pi}{4} + 2$, we have $f^{-1}(\frac{\pi}{4}+2) = 1$. (2%)
Then $\frac{d}{dx}f^{-1}\Big|_{\frac{\pi}{4}+2} = \frac{1}{f'(1)} = \frac{1}{\frac{1}{2}+2} = \frac{2}{5}$. (2%)

(c) The linear approximation of $f^{-1}(x)$ at $x = \frac{\pi}{4}+2$ is

$$L(x) = f^{-1}\left(\frac{\pi}{4}+2\right) + (f^{-1})'\left(\frac{\pi}{4}+2\right)\left(x - \frac{\pi}{4} - 2\right) = 1 + \frac{2}{5}\left(x - \frac{\pi}{4} - 2\right). \text{ (2\%)}$$

Then we estimate

$$f^{-1}\left(\frac{\pi}{4}+1.95\right) \approx L\left(\frac{\pi}{4}+1.95\right) = 1 + \frac{2}{5}(-0.05) = 0.98. \text{ (2\%)}$$

5. (13 pts) The minute hand on a clock is 6 cm long and the hour hand is 3 cm long. Let $\theta(t)$ be the angle between the minute hand and the hour hand at time t. Let $d(t)$ be the distance between the tips of the hands at time t.

(a) (2 pts) Find $\left|\frac{d\theta}{dt}\right|$ (rad/hour).

(b) (5 pts) Find $d'(t)$ in terms of θ.

(c) (6 pts) Find the maximum value of $d'(t)$.

Solution:

(a) (1 pt) $\begin{cases} \text{The minute hand rotates clockwise at a rate of one revolution per hour.} \\ \text{The hour hand rotates clockwise at a rate of } \frac{1}{12} \text{ revolution per hour.} \end{cases}$

(1 pt) Hence $\left|\frac{d\theta}{dt}\right| = (1 - \frac{1}{12}) \times 2\pi = \frac{11}{6}\pi$ (rad/hour)

(b) (2 pts: Formula for d^2) $d^2(t) = 3^2 + 6^2 - 2 \times 3 \times 6 \times \cos\theta(t) = 45 - 36\cos(\theta(t))$

(1 pt: 正確微分) $2d \cdot d'(t) = 36\sin(\theta(t)) \cdot \theta'(t)$

(2 pts: 帶入 d 和 $\frac{d\theta}{dt}$, 化簡)

$$\begin{cases} \text{Because } d(t) = \sqrt{45 - 36\cos\theta(t)}, \text{ we have } d'(t) = \dfrac{18\sin(\theta(t))}{\sqrt{45 - 36\cos\theta(t)}} \cdot \dfrac{d\theta}{dt} \\ \text{Hence } d'(t) = \pm 33\pi \dfrac{\sin(\theta(t))}{\sqrt{45 - 36\cos\theta(t)}} = \pm 11 \dfrac{\sin(\theta(t))}{\sqrt{5 - 4\cos\theta(t)}}\pi \text{ cm/hour} \end{cases}$$

(c) Find $\theta \in [0, 2\pi]$ such that $f(\theta) = \dfrac{\sin\theta}{\sqrt{5 - 4\cos\theta}}$ is maximized.

(2 pts: 正確微分)

$$f'(\theta) = \frac{\cos\theta}{\sqrt{5 - 4\cos\theta}} - \frac{\sin\theta \cdot 4\sin\theta}{2(5 - 4\cos\theta)^{3/2}} = \frac{1}{(5 - 4\cos\theta)^{3/2}}(5\cos\theta - 4\cos^2\theta - 2\sin^2\theta)$$

$$= \frac{1}{(5 - 4\cos\theta)^{3/2}}(5\cos\theta - 2\cos^2\theta - 2)$$

(1 pt: 因式分解)

$f'(\theta) = (5 - 4\cos\theta)^{-3/2}(2\cos\theta - 1)(2 - \cos\theta)$

$\because (5 - 4\cos\theta)^{-3/2} > 0$ and $(2 - \cos\theta) > 0$

(2 pts: 比較 critical number and endpoints 的函數值)

$\therefore f'(\theta) = 0$ if and only if $2\cos\theta - 1 = 0$ i.e. $\cos\theta = \frac{1}{2} \Leftrightarrow \theta = \frac{\pi}{3}$ or $\frac{5}{3}\pi$

$f(\frac{\pi}{3}) = \dfrac{\frac{\sqrt{3}}{2}}{\sqrt{3}} = \frac{1}{2}$, $f(\frac{5}{3}\pi) = -\frac{1}{2}$, $f(0) = 0$, $f(2\pi) = 0$

(1 pt) $\begin{cases} \text{Hence the maximum value of } f(\theta) \text{ on } [0, 2\pi] \text{ is } f(\frac{\pi}{3}) = \frac{1}{2}. \\ \text{The maximum } d'(t) \text{ is } \frac{11}{2}\pi \text{ (cm/hour).} \end{cases}$

6. (9 pts) Let $f(x)$ be a continuous function on $\mathbb{R}$. It is given that

$$\lim_{h\to 0}\frac{f(h)}{h} = 2020.$$

(a) (4 pts) Compute $f(0)$. Then, prove that f is differentiable at $x=0$ and compute $f'(0)$.

(b) (5 pts) Suppose in addition that f is twice differentiable and that $f''(x)\geq 2$ for all $x>0$. Using Mean Value Theorem, or otherwise, prove that

$$f(x)\geq 2020x + x^2 \text{ for all } x\geq 0.$$

Solution:

(a) (**Two points**)

(Method 1)

By limit laws,

$$\lim_{h\to 0} f(h) = \lim_{h\to 0}\frac{f(h)}{h}\cdot h = \lim_{h\to 0}\frac{f(h)}{h}\lim_{h\to 0} h = 2020\cdot 0 = 0.$$

Since f is continuous at 0, we have

$$f(0) = \lim_{h\to 0} f(h) = 0.$$

(Method 2)

Since f is continuous at 0, $\lim_{h\to 0} f(h) = f(0)$ exists. If $\lim_{h\to 0} f(h) = L \neq 0$, then $\lim_{h\to 0}\frac{f(h)}{h}$ does not exist, since $\lim_{h\to 0}|\frac{f(h)}{h}| = \infty$. Hence $\lim_{h\to 0} f(h) = f(0) = 0$.

(**Two points**)

$$f'(0) = \lim_{h\to 0}\frac{f(0+h)-f(0)}{h} \overset{f(0)=0}{=\!=\!=} \lim_{h\to 0}\frac{f(h)}{h} = 2020.$$

- "$\lim_{h\to 0} f(h)\neq 0$" includes the following cases:
  (a) $\lim_{h\to 0} f(h) = L\neq 0$
  (b) $\lim_{h\to 0} f(h)$ does not exist

  If one uses an argument like "if $\lim_{h\to 0} f(h)\neq 0$, then $\lim_{h\to 0} f(h)/h$ does not exist" without emphasising the existence of $\lim_{h\to 0} f(h)$, we are not sure whether he is aware of case (b).
- One can get the points of part 2 even he doesn't get any point of part 1.

(b) (**Two points**)(estimating f')

(Method 1)

Let $x>0$. Since f is twice-differentable, f' is continuous on $[0,x]$ and differentiable on $(0,x)$. By applying MVT to f', we have

$$\begin{aligned}\frac{f'(x)-f'(0)}{x-0} &= f''(c) \text{ for some } c\in(0,x)\\ &\geq 2.\end{aligned}$$

Hence

$$f'(x)\geq 2020+2x.$$

(Method 2)

Racetrack Principle. *Suppose both F and G are differentiable functions. If $F(0) = G(0)$ and $F'(x) \geq G'(x)$ for all $x > 0$, then $F(x) \geq G(x)$ for all $x > 0$.*

Proof. Let $H(x) = F(x) - G(x)$. For all $x > 0$, H is continuous on $[0, x]$ and differentiable on $(0, x)$. By MVT,

$$\begin{aligned}\frac{H(x) - H(0)}{x - 0} \overset{H(0)=0}{=\!=\!=} \frac{H(x)}{x} &= H'(c) \text{ for some } c \in (0, x) \\ &= F'(c) - G'(c) \geq 0.\end{aligned}$$

Hence $H(x) = xH'(c) \geq 0$, which implies $F(x) \geq G(x)$ for $x > 0$. □

Let $F(x) = f'(x)$ and $G(x) = 2x + 2020$. Then $F(0) = f'(0) = 2020 = G(0)$ and $F'(x) = f''(x) \geq 2 = G'(x)$ for $x > 0$. Hence by Racetrack Principle we have $f'(x) \geq 2x + 2020$ for $x > 0$.

(**Three points**)(estimating f)
(Method 1)
Let $g(x) = f(x) - (2020x + x^2)$. By MVT, for all $x > 0$,

$$\frac{g(x) - g(0)}{x - 0} \overset{f(0)=0}{=\!=\!=} \frac{g(x)}{x} = g'(d) \text{ for some } d \in (0, x).$$

Now

$$g'(d) = f'(d) - (2020 + 2d) \geq 0$$

from the previous part. Hence

$$f(x) - (2020x + x^2) = g(x) = xg'(d) \geq 0 \Rightarrow f(x) \geq 2020x + x^2$$

for all $x > 0$. When $x = 0$ the inequality is obviously true.

(Method 2)
Let $F(x) = f(x)$ and $G(x) = 2020x + x^2$. Then $F(0) = 0 = G(0)$ and $F'(x) = f'(x) \geq 2020 + 2x = G'(x)$ for all $x > 0$. By Racetrack Principle, we have $f(x) \geq 2020x + x^2$ for all $x > 0$. When $x = 0$ the inequality is obviously true.

(Method 3)

$$f(x) \overset{f(0)=0}{=\!=\!=} f(x) - f(0) = \int_0^x f'(t)dt \geq \int_0^x (2020 + 2t)dt = 2020x + x^2.$$

- Any argument using integration of f'' will not get any point of the first part (estimating f'). There exists a differentiable function h such that h' is not Riemann-integrable and is bounded below.
- Since the Racetrack Principle is not contained in the textbook, one needs to state and prove the theorem to get all points. Without stating and proving the theorem, if no mistake is made in using the theorem, one point is awarded for each part (totally two points).
- If one uses integration argument not correctly in the second part (using indefinite integral, using vague word "by integration", etc.), one point is awarded.

7. (13 pts) Suppose that the <u>derivative</u> of the function f is given,

$$\frac{d}{dx}f(x) = \sqrt[3]{x(9+x)(9-x)}.$$

(a) (3 pts) Find the critical numbers of f.

(b) (2 pts) Find the intervals on which f is decreasing.

(c) (4 pts) Find the intervals on which f is concave upward.

(d) (4 pts) Sketch the curve $y = f(x)$ assuming $f(0) = 0$. The sketch just needs to capture increase/decrease and concavity of the function.

Solution:

(a) The critical numbers are the x-values when $f'(x) = 0$ or when $f'(x)$ is not defined. The derivative $f'(x)$ has domain $\mathbb{R}$ hence the critical numbers of f are the solutions of $f'(x) = 0$.

$$f'(x) = 0$$

$$\sqrt[3]{x(9+x)(9-x)} = 0$$

$$x(9+x)(9-x) = 0$$

The critical numbers of f are $x = -9, 0, 9$. □

(b) We know that the intervals where $f'(x) < 0$ are intervals of decrease. First we solve the inequality $f'(x) < 0$.

$$\sqrt[3]{x(9+x)(9-x)} < 0$$

$$x(9+x)(9-x) < 0$$

The intervals where $f"(x) < 0$ are intervals when all three terms are negative or only $(9-x) < 0$, hence $(-9,0)$ and $(9,\infty)$ are intervals of decrease for f. □

Note: To be careful we should check the endpoints $x = -9, 0, 9$ in addition to the intervals when $f'(x) < 0$. The definition of an interval of decrease for f is given by

$$f(x_1) > f(x_2) \text{ whenever } x_1 < x_2 \text{ and } x_1, x_2 \text{ are in the interval.}$$

In order to check the endpoints, we need to verify the following 3 statements:

(1) $f(-9) > f(x)$ for all $x \in (-9, 0)$.
(2) $f(0) < f(x)$ for all $x \in (-9, 0)$.
(3) $f(9) > f(x)$ for all $x \in (9, \infty)$.

They can be easily proved using the Mean Value Theorem. The intervals $[-9, 0]$ and $[9, \infty)$ are maximal intervals of decrease for f. □

(c) We know that the intervals where $f''(x) > 0$ are intervals of upward concavity. The intervals with upward concavity can be found by taking the derivative of $f'(x)$ and solving the inequality $f''(x) > 0$.

$$f''(x) = \frac{d}{dx}\left[(81x - x^3)^{(1/3)}\right] = \frac{1}{3}(81x - x^3)^{(-2/3)}(81 - 3x^2) = \frac{27 - x^2}{\sqrt[3]{(x(9+x)(9-x))^2}}$$

Solve $f''(x) = 0$ to get $x = \pm 3\sqrt{3}$. The second derivative is not defined at $x = -9, 0, 9$. Notice that term $(81x - x^3)^{(-2/3)}$ is always positive. Therefore

$$f''(x) > 0$$

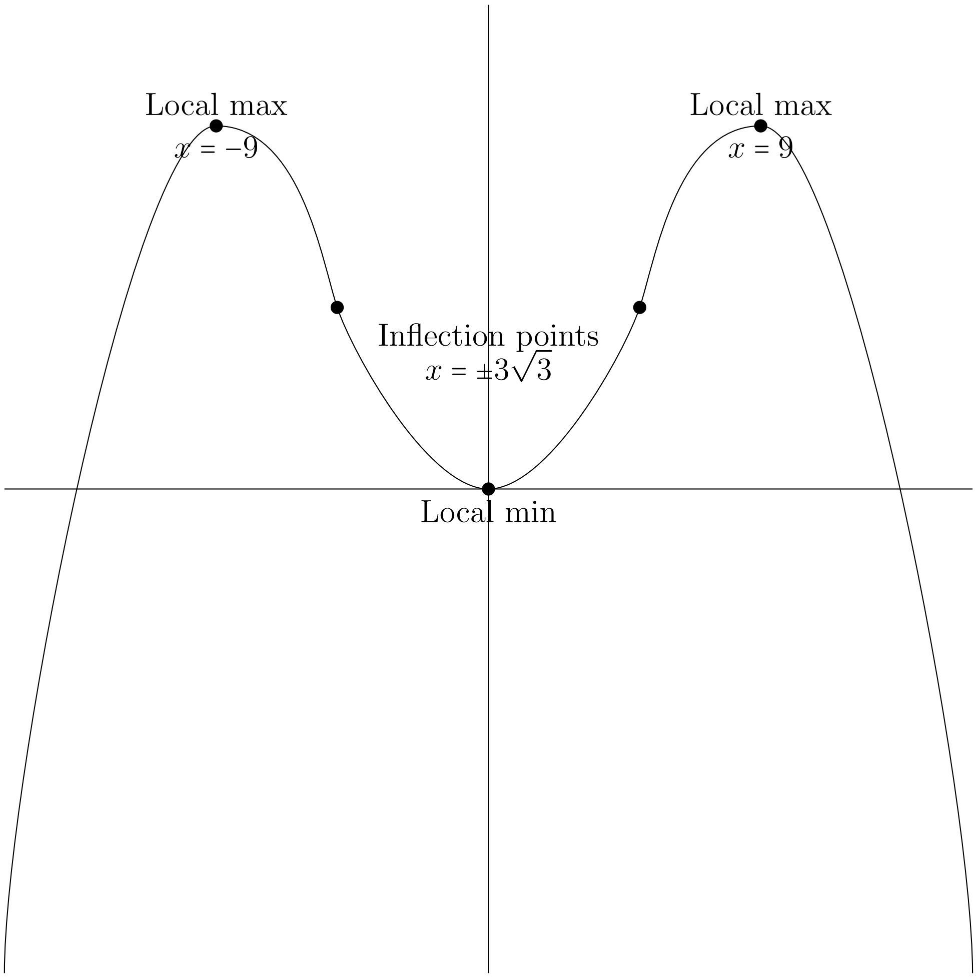
$$27 - x^2 > 0$$
$$|x| < \sqrt{27}$$

The interval with upward concavity is $(-3\sqrt{3}, 0)$ and $(0, 3\sqrt{3})$. □

Note: To be careful we should check the endpoints $x = -3\sqrt{3}, 0, 3\sqrt{3}$ in addition to the intervals when $f''(x) > 0$. The definition of concavity requires us to compare the tangent line and the graph of the function.

The 2 endpoints $x = \pm 3\sqrt{3}$ correspond to inflections of the graph, hence f is not concave up at $x = \pm 3\sqrt{3}$. Even though $f''(0)$ is not defined, we see that f is differentiable at $x = 0$ and $f'(0) = 0$. The tangent line of f at $x = 0$ is a horizontal tangent line. From (b) we can see that f is decreasing on $(-3\sqrt{3}, 0)$ and increasing on $(0, 3\sqrt{3})$. Therefore the graph of f is above the tangent line near $x = 0$, meaning that f has upward concavity at $x = 0$. The maximal interval of upward concavity is $(-3\sqrt{3}, 3\sqrt{3})$. □

(d) An example sketch that shows increasing, decreasing, and concavity:



□

Grading scheme: we subtract points for mistakes until it reaches zero.

7(a) (3 pts) 1 point off for **each** missing or extra critical number. If they display **clear** errors in solving for critical numbers, 1 point off for each math error but they get points for the answer. It needs to be the x-values.

7(b) (2 pts) Depends on the answer in (a). 1 point off if the interval answer doesn't have the same endpoints as their answer in (a). 1 point off for **each** missing or extra interval. If they display **clear** errors in solving the inequality, 1 point off for **each** math error but they get points for the answer.

For using closed intervals in (b) or (c) will be 1 point off just once. *Note: later discussions resulted in treating closed intervals as correct answers as well but students need to show more work. For future reference we should not take points off for closed intervals unless the function is not defined at the endpoint.*

7(c) (4 pts) Break this problem into 2 steps:

Step 1: (1 pts) Find and factor the second derivative. (no points off if a calculation mistake doesn't change any signs to the second derivative.)

Step 2: (3 pts) Depends on the answer in step 1. 1 point off for **each** missing or extra interval. If they display **clear** errors in solving the inequality (main example: not noticing $(81x - x^3)^{2/3}$ is always positive), 1 point off for **each** math error but they get points for the answer.

Note: later discussions resulted in giving points to some cases of including $x = 0$ *in the answer even without a detailed explanation.*

7(d) (4 pts) Depends on their answers in (b) and (c). 1 point off if the sketch uses points not in their answers in (b) and (c) (or the sketch did not label the x-values). 1 point off if the sketch is not differentiable. 0.5 points off if the curve doesn't go through $(0,0)$.

1 point off for **each** interval showing the wrong inc/dec and concavity (it is okay if concavity is not accurate as long as they show it clearly in the middle of the interval).

If student read the problem wrong, 1 point off at the start. then use the same grading scheme. Only exception here is if the student miscopy or read the problem wrong and simplified the problem (example: $f'(x) = x(9+x)(9-x)$ or $f(x) = x(9+x)(9-x)$), in these cases 3 points off at the start.

Examples so far:

(1) Intervals of concavity including undefined values without checking (see note in solution). 1 point off for each.

(2) Upside down graph: 1 point off if a **clear** sign mistake is found. Otherwise no points for the graph.

(3) No work for many parts: for (a) and (b) it is fine to write answers without work. For (c), no work means no points. (d) depends on (b) and (c) so no points for (d) if they didn't show their answers for (b) and (c).

(4) Non-smooth sketch: for students with mistakes in (b) and (c) it is likely their sketch will be non-smooth. 1 point off for not differentiable.

(Just in case students read the problem wrong) Solution:

$$f(x) = (81x - x^3)^{(1/3)}$$

$$f'(x) = \frac{1}{3}(81x - x^3)^{(-2/3)}(81 - 3x^2) = \frac{27 - x^2}{\sqrt[3]{(81x - x^3)^2}}$$

$$f''(x) = \frac{-2x}{\sqrt[3]{(81x - x^3)^2}} - \frac{2(27 - x^2)^2}{\sqrt[3]{(81x - x^3)^5}}$$

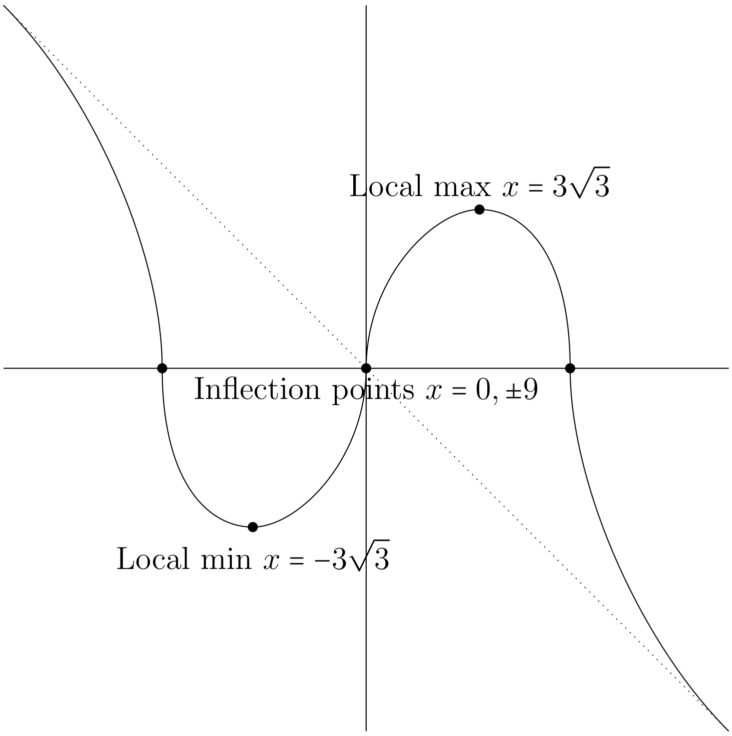
$$= \frac{2x^4 - 162x^2 - 2x^4 + 108x^2 - 1458}{\sqrt[3]{(81x - x^3)^5}} = \frac{-54(x^2 + 27)}{\sqrt[3]{x^5(9 - x)^5(9 + x)^5}}$$

(a) Critical numbers are $x = 0, \pm 3\sqrt{3}, \pm 9$.

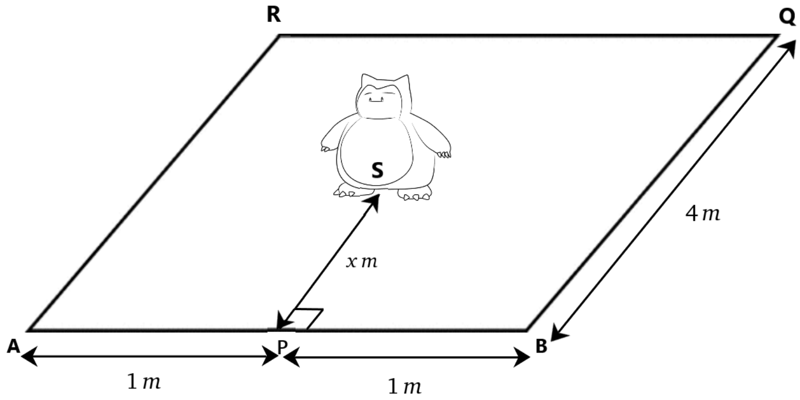
(b) The decreasing intervals are $(-\infty, -9), (-9, -3\sqrt{3}), (3\sqrt{3}, 9), (9, \infty)$. (Or after checking, $(-\infty, -3\sqrt{3}), (3\sqrt{3}, \infty)$.)

(c) The intervals of upward concavity $(-9, 0), (9, \infty)$

(d) The sketch has slanted asymptotes $y = -x$ in both directions. Vertical tangent lines at $(-9, 0), (0, 0), (9, 0)$.



8. (13 pts) Three trainers P, Q, R have spotted Snorlax(卡比獸) in a rectangular field (See figure below). Trainer P is standing in the middle of the edge $\overline{AB}$ whereas Trainers Q and R are at the two corners of the field opposite to $\overline{AB}$. Snorlax is currently at position S which is at a distance of x meters in front of trainer P.



(a) (3 pts) Let L be the total distances of the three trainers from Snorlax. Prove that

$$\frac{dL}{dx} = 1 - \frac{2(4-x)}{\sqrt{(4-x)^2+1}}.$$

(b) (6 pts) Find the values of x at which L attains its greatest and least value respectively.

(c) (4 pts) Snorlax is now asleep. To wake Snorlax up, each trainer is going to play a magical flute. The intensity of sound energy received by Snorlax from each flute varies inversely with the square of the distance from the flute. In other words, if E is the total sound energy received by Snorlax from the three flutes, then

$$E = \frac{k}{\overline{PS}^2} + \frac{k}{\overline{QS}^2} + \frac{k}{\overline{RS}^2} \quad \text{for some constant } k > 0.$$

Trainer P claims that when L attains the least value, E will attain its greatest value. Do you agree with trainer P? Explain your answer.

Solution:

(a) By the Pythagoras Theorem, we have

$$QS = RS = \sqrt{(4-x)^2 + 1^2}. \qquad (1M)$$

Therefore, $L = PS + QS + RS = x + 2\sqrt{(4-x)^2+3}$ and hence

$$\begin{aligned} \frac{dL}{dx} &= 1 + 2 \cdot \frac{1}{2} \cdot \frac{1}{\sqrt{(4-x)^2+1}} \cdot (2(4-x)) \cdot (-1) \qquad (1+1M) \\ &= 1 - \frac{2(4-x)}{\sqrt{(4-x)^2+1}}. \end{aligned}$$

Marking Scheme :
1M for correctly writing down the length of QS (or RS)
1M for the attempt to differentiate L by the chain rule (not necessarily correctly)
1M for differentiating L correctly

(b) To compute the critical points, we set $\frac{dL}{dx} = 0$ (1 M). Therefore we have

$$\begin{aligned} 2(4-x) &= \sqrt{(4-x)^2+1} \\ \Rightarrow 4(4-x)^2 &= (4-x)^2+1 \\ \Rightarrow 3(4-x)^2 &= 1 \\ \Rightarrow x = 4-\frac{1}{\sqrt{3}} &\text{ or } x = 4+\frac{1}{\sqrt{3}} \text{ (rejected, as } 0 \le x \le 4).(1M) \end{aligned}$$

x	0	$\cdots$	$4-\frac{1}{\sqrt{3}}$	$\cdots$	4
$\frac{dL}{dx}$	$-$	$-$	0	$+$	$+$
L	$2\sqrt{17}$	$\searrow$	min	$\nearrow$	6

(1M)

Therefore, L attains its minimum value at $x = 4-\frac{1}{\sqrt{3}}$. (1M)

Since $L(0) = 2\sqrt{17} > 2 \cdot 4 = 8$ and $L(4) = 6$ (1M), we have $L(0) > L(4)$ and hence L attains its maximum value at $x = 0$. (1M)

Marking Scheme :
1M for setting $\frac{dL}{dx} = 0$ (or equivalent)
1M for the correct critical number (-0.5M for forgetting to reject $x = 4+\frac{1}{\sqrt{3}}$)
1M for an attempt to use 1st or 2nd derivative tests (not necessarily correctly)
1M for any correct argument that x attains a minimum at $x = 4-\frac{1}{\sqrt{3}}$
(either by 1st or 2nd derivative tests)
1M for computing $L(0)$ and $L(4)$
1M for any correct argument that L attains its maximum at $x = 0$

(c) From the given information, we have

$$E = \frac{k}{x^2} + \frac{2k}{(4-x)^2+1} \text{ for some constant } k > 0.(1M)$$

Differentiating with respect to x gives

$$\frac{dE}{dx} = k\left(-\frac{2}{x^3} + \frac{4(4-x)}{((4-x)^2+1)^2}\right).(1M)$$

By (b), L attains its minimum value at $x = 4-\frac{1}{\sqrt{3}}$. However,

$$\left.\frac{dE}{dx}\right|_{x=4-3^{-\frac{1}{2}}} = k\left(-\frac{2}{\left(4-3^{-\frac{1}{2}}\right)^3} + \frac{9}{4\sqrt{3}}\right) > 0.$$

Therefore, E is strictly increasing at $x = 4-\frac{1}{\sqrt{3}}$ and does not attain its maximum value there. (1M) Hence Trainer P's claim is incorrect. (1M)

Marking Scheme :
1M for writing down a formula for E or an attempt to differentiate E
1M for the correct formula for dE/dx
1M for any correct argument that E does not attain its maximum value when $x = 4 - \dfrac{1}{\sqrt{3}}$
(either by 1^{st} or 2^{nd} derivative tests)
1M for rejecting the claim of Trainer P (with or without the correct reasoning)