1. (6%) Find $f'\left(\frac{\pi}{4}\right)$ if $f(x) = e^{g(x)}$ and $g(x) = \int_1^{\tan x} \sqrt{1+t^3}\, \mathrm{d}t$.

Solution:
We have $f'(x) = e^{g(x)} g'(x)$(1 point). From Fundamental Theorem of Calculus (1 point),

$$g'(x) = \sqrt{1+\tan^3 x} \cdot \sec^2 x \text{(2 point)}.$$

We obtain that

$$e^{g(\pi/4)} = 1\text{(1 point) and } g'(\frac{\pi}{4}) = \sqrt{1+\tan^3(\pi/4)}\sec^2(\pi/4) = 2\sqrt{2}\text{(1 point)}.$$

Therefore $f'(\pi/4) = 2\sqrt{2}$.

2. (14%) Compute the following integrals.

(a) (7%) $\int (\cos x + \sec^2 x) \ln(\tan x)\,\mathrm{d}x$.

(b) (7%) $\int x\sqrt{2x - x^2}\,\mathrm{d}x$

Solution:

- One point is deducted if the integration constant is missing.
- For each integration, one point is deducted if the answer is correct except the sign.

(a) Method 1 **(2+2+2+1)**

$$\int (\cos x + \sec^2 x) \ln(\tan x)\,dx = \int \cos x \ln(\tan x)\,dx + \int \sec^2 x \ln(\tan x)\,dx$$

$$\begin{aligned}\int \cos x \ln(\tan x)\,dx &= \int \ln(\tan x)\,d(\sin x) \\ &= \sin x \ln(\tan x) - \int \sin x\, d[\ln(\tan x)] \\ &= \sin x \ln(\tan x) - \int \sin x \cdot \frac{\sec^2 x}{\tan x}\,dx \\ &= \sin x \ln(\tan x) - \int \sec x\,dx \quad \textbf{(2 points)} \\ &= \sin x \ln(\tan x) - \ln|\tan x + \sec x| + C_1 \quad \textbf{(2 points)}\end{aligned}$$

$$\begin{aligned}\int \sec^2 x \ln(\tan x)\,dx \overset{u=\tan x}{=\!=\!=} & \int \ln u\,du \\ &= u \ln u - \int u\,d(\ln u) \\ &= u \ln u - \int 1\,du) \\ &= u \ln u - u + C_2 \\ &= \tan x \ln(\tan x) - \tan x + C_2 \quad \textbf{(2 points)}\end{aligned}$$

Hence

$$\int (\cos x + \sec^2 x) \ln(\tan x)\,dx = (\sin x + \tan x) \ln(\tan x) - \ln|\tan x + \sec x| - \tan x + C \quad \textbf{(1 point)}$$

Method 2 **(2+2+2+1)**

$$\begin{aligned}&\int (\cos x + \sec^2 x) \ln(\tan x)\,dx \\ &= \int \ln(\tan x)\,d(\sin x + \tan x) \\ &= (\sin x + \tan x) \ln(\tan x) - \int (\sin x + \tan x)\,d(\ln(\tan x)) \\ &= (\sin x + \tan x) \ln(\tan x) - \int (\sec x + \sec^2 x)\,dx = I \quad \textbf{(2 points)}\end{aligned}$$

Since

$$\int \sec x\,dx = \ln|\tan x + \sec x| + C_1 \quad \textbf{(2 points)}$$

and

$$\int \sec^2 x\,dx = \tan x + C_2, \quad \textbf{(2 points)}$$

we have

$$I = (\sin x + \tan x) \ln(\tan x) - \ln|\tan x + \sec x| - \tan x + C \quad \textbf{(1 point)}$$

(b) Method 1 **(2+1+1+1+1+1)**

$$\int x\sqrt{2x-x^2}\,dx = \int x\sqrt{1-(x-1)^2}\,dx \overset{x-1=\sin\theta}{=\!=\!=} \int (\sin\theta+1)\cos\theta(\cos\theta\,d\theta)$$
$$= \int \sin\theta\cos^2\theta\,d\theta + \int \cos^2\theta\,d\theta \quad \textbf{(2 points)}$$
$$= I_1 + I_2$$

$$I_1 = \int \sin\theta\cos^2\theta\,d\theta = -\int \cos^2\theta\,d(\cos\theta)$$
$$= -\frac{\cos^3\theta}{3} + C_1 \quad \textbf{(1 point)}$$

$$\cos\theta = \sqrt{1-\sin^2\theta} = \sqrt{1-(x-1)^2} = \sqrt{2x-x^2}$$
$$\Rightarrow I_1 = -\frac{2x-x^2}{3}\sqrt{2x-x^2} + C_1 \quad \textbf{(1 point)}$$

$$I_2 = \int \cos^2\theta\,d\theta = \int \frac{1+\cos 2\theta}{2}\,d\theta$$
$$= \frac{\theta}{2} + \frac{\sin 2\theta}{4} + C_2 \quad \textbf{(1 point)}$$

$$\sin 2\theta = 2\sin\theta\cos\theta = 2(x-1)\sqrt{2x-x^2}$$
$$\Rightarrow I_2 = \frac{\sin^{-1}(x-1)}{2} + \frac{(x-1)\sqrt{2x-x^2}}{2} + C_2 \quad \textbf{(1 point)}$$

Hence

$$\int x\sqrt{2x-x^2}\,dx = [-\frac{2x-x^2}{3}\sqrt{2x-x^2} + C_1]$$
$$+ [\frac{\sin^{-1}(x-1)}{2} + \frac{(x-1)\sqrt{2x-x^2}}{2} + C_2]$$
$$= \frac{2x^2-x-3}{6}\sqrt{2x-x^2} + \frac{\sin^{-1}(x-1)}{2} + C \quad \textbf{(1 point)}$$

Method 2 **(2+1+1+1+1+1)**

$$\int x\sqrt{2x-x^2}\,dx = \int (x-1)\sqrt{2x-x^2}\,dx + \int \sqrt{2x-x^2}\,dx \quad \textbf{(2 points)}$$
$$= I_1 + I_2$$

$$I_1 = \int (x-1)\sqrt{2x-x^2}\,dx \overset{u=2x-x^2}{=\!=\!=} \frac{-1}{2}\int \sqrt{u}\,du$$
$$= \frac{-1}{3}u^{3/2} + C_1 \quad \textbf{(1 point)}$$
$$= -\frac{2x-x^2}{3}\sqrt{2x-x^2} + C_1 \quad \textbf{(1 point)}$$

$$I_2 = \int \sqrt{2x-x^2}\,dx = \int \sqrt{1-(x-1)^2}\,dx \overset{x-1=\sin\theta}{=\!=\!=} \int \cos\theta(\cos\theta\,d\theta)$$
$$= \int \cos^2\theta\,d\theta = \int \frac{1+\cos 2\theta}{2}$$
$$= \frac{\theta}{2} + \frac{\sin 2\theta}{4} + C_2 \quad \textbf{(1 point)}$$

$$\sin 2\theta = 2\sin\theta\cos\theta = 2(x-1)\sqrt{2x-x^2}$$
$$\Rightarrow I_2 = \frac{\sin^{-1}(x-1)}{2} + \frac{(x-1)\sqrt{2x-x^2}}{2} + C_2 \quad \textbf{(1 point)}$$

Hence

$$\int x\sqrt{2x-x^2}\,dx = [-\frac{2x-x^2}{3}\sqrt{2x-x^2}+C_1] + [\frac{\sin^{-1}(x-1)}{2}+\frac{(x-1)\sqrt{2x-x^2}}{2}+C_2]$$
$$= \frac{2x^2-x-3}{6}\sqrt{2x-x^2}+\frac{\sin^{-1}(x-1)}{2}+C \quad \textbf{(1 point)}$$

(Alternative calculation of I_2)
Let $u=\sqrt{x}$. Then

$$\begin{aligned}
I_2 &= \int \sqrt{2u^2-u^4}\,2u\,du \\
&= 2\int u^2\sqrt{2-u^2}\,du \\
&= 2\int 2\sin^2\theta\cdot\sqrt{2}\cos\theta\cdot\sqrt{2}\cos\theta\,d\theta && (\text{let } u=\sqrt{2}\sin\theta) \\
&= 8\int \sin^2\theta\cos^2\theta\,d\theta \\
&= 2\int \sin^2 2\theta\,d\theta \\
&= \int (1-\cos 4\theta)\,d\theta = \theta - \frac{\sin 4\theta}{4}+C_2 \quad \textbf{(1 point)}
\end{aligned}$$

$$\sin 4\theta = 2\sin 2\theta\cos 2\theta = 4\sin\theta\cos\theta(2\cos^2\theta-1) = 2(1-x)\sqrt{2x-x^2}$$
$$\Rightarrow I_2 = \sin^{-1}\sqrt{\frac{x}{2}}+\frac{(x-1)\sqrt{2x-x^2}}{2}+C_2 \quad \textbf{(1 point)}$$

Note that $\sin^{-1}\sqrt{\frac{x}{2}}$ and $\frac{\sin^{-1}(x-1)}{2}$ differ by a constant.

3. (a) (7%) Evaluate the integral $\int \frac{3x^2 - x - 5}{(x-2)(x^2+1)}\,\mathrm{d}x$.

(b) (5%) Using the substitution $u = \frac{1}{x}$, show that for any $a > 1$, $\int_{1/a}^{a} \frac{\ln x}{1 + x + x^2}\,\mathrm{d}x = 0$.

Solution:

(a) Since

$$\frac{3x^2 - x - 5}{x^3 - 2x^2 + x - 2} = \frac{1}{x-2} + \frac{2x+3}{x^2+1}. \qquad \text{(3 points)}$$

That is, two points for being able to work out partial fractions.

$$= \frac{1}{x-2} + \frac{2x}{x^2+1} + \frac{3}{x^2+1} \qquad \text{(+1 point)}$$

That is, two points for able to realize the integral $\frac{2x+3}{x^2+1}$ need some extra care.

So

$$\int \left(\frac{1}{x-2} + \frac{2x+3}{x^2+1}\right)dx = \ln|x-2| + \ln|x^2+1| + 3\tan^{-1} x + C.$$

(+1 point for each integral)

For example, if the answer is $\ln|x-2| + \int \frac{2x+3}{x^2+1}dx$. Then it receives $3 + 1 = 4$ points.

(b) Since $x = \frac{1}{u}$, $dx = \frac{-1}{u^2}du$ (1 point)

Therefore, $\frac{\ln x}{1+x+x^2} = \frac{-u^2 \ln u}{1+u+u^2}$ (2 points)

$$\int_{1/a}^{a} \frac{\ln x}{1+x+x^2}dx = \int_{a}^{1/a} \frac{\ln u}{1+u+u^2}du.$$

The correct integration limits worth for 1 point

Hence $I = -I$ and therefore $I = 0$. (1 point). That is, being able to conclude the argument get 1 point.

4. Determine whether the improper integral converges. If it converges, compute the value.

(a) (6%) $\displaystyle\int_0^1 \frac{\mathrm{d}t}{\sqrt{t}(1+t)}$. (b) (6%) $\displaystyle\int_0^\infty \frac{\mathrm{d}t}{\sqrt{t}(1+t)}$.

Solution:

(a) Since the infinite discontinuity occurs at the left endpoint of $[0,1]$, we have

$$\int_0^1 \frac{dt}{\sqrt{t}(1+t)} = \lim_{a\to 0^+}\int_a^1 \frac{dt}{\sqrt{t}(1+t)}. \quad (1\%)$$

Set $u=\sqrt{t}$. We have $dt = 2u\,du$. When $t=a$, $u=\sqrt{a}$. When $t=1$, $u=1$. (2%)
Then

$$\begin{aligned}\int_a^1 \frac{dt}{\sqrt{t}(1+t)} &= \int_{\sqrt{a}}^1 \frac{2u\,du}{u(1+u^2)} = \int_{\sqrt{a}}^1 \frac{2\,du}{(1+u^2)} = 2\tan^{-1}u\Big]_{\sqrt{a}}^1 \\ &= 2(\tan^{-1}1-\tan^{-1}\sqrt{a}) = \frac{\pi}{2}-2\tan^{-1}\sqrt{a} \quad (2\%)\end{aligned}$$

Therefore,

$$\int_0^1 \frac{dt}{\sqrt{t}(1+t)} = \lim_{a\to 0^+}\int_a^1 \frac{dt}{\sqrt{t}(1+t)} = \lim_{a\to 0^+}\left(\frac{\pi}{2}-2\tan^{-1}\sqrt{a}\right) = \frac{\pi}{2}. \quad (1\%)$$

(b)

$$\begin{aligned}\int_0^\infty \frac{dt}{\sqrt{t}(1+t)} &= \int_0^1 \frac{dt}{\sqrt{t}(1+t)} + \int_1^\infty \frac{dt}{\sqrt{t}(1+t)} \quad (1\%) \\ &= \lim_{a\to 0^+}\int_a^1 \frac{dt}{\sqrt{t}(1+t)} + \lim_{b\to\infty}\int_1^b \frac{dt}{\sqrt{t}(1+t)} \quad (1\%)\end{aligned}$$

Set $u=\sqrt{t}$. We have $dt=2u\,du$. When $t=1$, $u=1$. When $t=b$, $u=\sqrt{b}$. (1%)
Then

$$\begin{aligned}\int_1^b \frac{dt}{\sqrt{t}(1+t)} &= \int_1^{\sqrt{b}} \frac{2u\,du}{u(1+u^2)} = \int_1^{\sqrt{b}} \frac{2\,du}{(1+u^2)} = 2\tan^{-1}u\Big]_1^{\sqrt{b}} \\ &= 2(\tan^{-1}\sqrt{b}-\tan^{-1}1) = 2\tan^{-1}\sqrt{b}-\frac{\pi}{2} \quad (1\%)\end{aligned}$$

So we have

$$\int_1^\infty \frac{dt}{\sqrt{t}(1+t)} = \lim_{b\to\infty}\int_1^b \frac{dt}{\sqrt{t}(1+t)} = \lim_{b\to\infty}\left(2\tan^{-1}\sqrt{b}-\frac{\pi}{2}\right) = \frac{\pi}{2}. \quad (1\%)$$

Therefore, from (a) and the result shown in the above, we obtain

$$\int_0^\infty \frac{dt}{\sqrt{t}(1+t)} = \int_0^1 \frac{dt}{\sqrt{t}(1+t)} + \int_1^\infty \frac{dt}{\sqrt{t}(1+t)} = \frac{\pi}{2}+\frac{\pi}{2} = \pi \quad (1\%)$$

5. (12%) Let C be the curve defined by $y = f(x) = \int_e^x \sqrt{(\ln t)^2 - 1}\,\mathrm{d}t$, $e \le x \le e^2$.

(a) (6%) Find the arc length of the curve C.

(b) (6%) Find the area of the surface generated by rotating C about the y-axis.

Solution:

(a) By the fundamental theorem of calculus, we have $\frac{dy}{dx} = \sqrt{(\ln x)^2 - 1}$. (2 pts)

and

$$\sqrt{1 + (\frac{dy}{dx})^2} = \sqrt{1 + (\sqrt{(\ln x)^2 - 1})^2} = \sqrt{1 + (\ln x)^2 - 1} = \ln x.$$

So

$$\text{arc length of } C = \int_e^{e^2} \sqrt{1 + (\frac{dy}{dx})^2} dx = \int_e^{e^2} \ln x dx \qquad \text{(2 pts)}$$

$$= (x \cdot \ln x - x) \Big|_e^{e^2} = x \cdot (\ln x - 1) \Big|_e^{e^2} = e^2. \qquad \text{(2 pts)}$$

(b)

$$\text{area of the surface} = \int_e^{e^2} 2\pi x \cdot \sqrt{1 + (\frac{dy}{dx})^2} dx = 2\pi \cdot \int_e^{e^2} x \cdot \ln x dx \qquad \text{(3 pts)}$$

$$= \pi \cdot \int_e^{e^2} \ln x (dx^2) = \pi \cdot [(x^2 \cdot \ln x) \Big|_e^{e^2} - \int_e^{e^2} x^2 \cdot \frac{1}{x} dx]$$

$$= \pi \cdot [2e^4 - e^2 - \int_e^{e^2} x dx] = \pi \cdot [2e^4 - e^2 - \frac{x^2}{2} \Big|_e^{e^2}]$$

$$= \pi \cdot [2e^4 - e^2 - \frac{1}{2}(e^4 - e^2)] = \frac{\pi}{2}(3e^4 - e^2). \qquad \text{(3 pts)}$$

(電機)

(a) (8%) Find the general solution of $y''(t) + y(t) = \sec^2 t$, for $-\frac{\pi}{2} < t < \frac{\pi}{2}$.

(b) (8%) Solve the differential equation $y''(t) + y(t) = f(t)$, $y(0) = 0$, $y'(0) = 1$, for $t \geq 0$, where $f(t) = t - 5$ for $5 \leq t < 10$, $f(t) = 0$, otherwise. (Hint: $\mathcal{L}\{g(t-a)\mathcal{U}(t-a)\} = e^{-as}\mathcal{L}\{g(t)\}$)

Solution:

(a) The auxiliary equation of the differential equation $y'' + y = 0$ is $r^2 + 1 = 0$, which ha solutions $r = \pm i$. Hence the general soltuion of $y'' + y = 0$ is $C_1 \cos t + C_2 \sin t$, where C_1, C_2 are constants. (1pt)
Now we try a particular solution of $y'' + y = \sec^2 t$ of the forms $y_p = u_1(t)\cos t + u_2(t)\sin t$ ⋯(1pt)
Impose condition $u_1'(t)\cos t + u_2'(t)\sin t = 0$. Then $y_p'' + y_p = -u_1(t)\sin t + u_2(t)\cos t = \sec^2 t$ ⋯(2pts)

$$\begin{cases} u_1'(t)\cos t + u_2'(t)\sin t = 0 \\ -u_1(t)\sin t + u_2(t)\cos t = \sec^2 t \end{cases} \Rightarrow \begin{array}{l} u_1'(t) = -\sec^2 t \sin t \\ u_2'(t) = \sec t \end{array} \quad \cdots\cdots\cdots\cdots(1\text{pt})$$

$$\left.\begin{array}{l} \Rightarrow u_1(t) = -\displaystyle\int \sec^2 t \sin t\,dt = -\int \sec t \tan t\,dt = \sec t + C \\ \quad u_2(t) = \displaystyle\int \sec t\,dt = \ln|\sec t + \tan t| + C \\ \text{Hence a particular solution is} \\ y_p = -\sec t \cos t + (\ln|\sec t + \tan t|)\sin t = -1 + \sin t(\ln|\sec t + \tan t|) \end{array}\right\} \cdots\cdots\cdots\cdots(2\text{pts})$$

The general solution o f $y'' + y = sec^2 t$ is

$$-1 + (\sin t)\ln|\sec t + \tan t| + C_1 \cos t + C_2 \sin t,$$

where C_1, C_2 are constant. (For $-\frac{\pi}{2} < t < \frac{\pi}{2}$, $\sec t + \tan t > 0$,
the solution can be written as $-1 + (\sin t)\ln(\sec t + \tan t) + C_1 \cos t + C_2 \sin t)$⋯⋯⋯⋯(1pt)

(b) $y''(t) + y(t) = (t-5)(\mathscr{U}(t-5) - \mathscr{U}(t-10))$, where $\mathscr{U}(t-a) = \begin{cases} 0, & 0 \leq t < a \\ 1, & t \geq a \end{cases}$
$= (t-5)\mathscr{U}(t-5) - (t-10)\mathscr{U}(t-10) - 5\mathscr{U}(t-10)$ ⋯⋯⋯⋯(1pt)
Apply Laplace transform on the differential equation. We obtain

$$(s^2 Y(s) - sy(0) - y'(0)) + Y(s) = e^{-5s}\frac{1}{s^2} - e^{-10s}\frac{1}{s^2} - 5e^{-10s}\frac{1}{s}$$

(1pt for $\mathscr{L}\{y''(t)\}$, $\mathscr{L}\{t\}$ and $\mathscr{L}\{1\}$)

$$\left.\begin{array}{l} (s^2+1)Y(s) = 1 + \dfrac{1}{s^2}e^{-5s} - \dfrac{1}{s^2}e^{-10s} - \dfrac{5}{s}e^{-10s} \\ \Rightarrow Y(s) = \dfrac{1}{s^2+1} + \dfrac{1}{s^2(s^2+1)}(e^{-5s} - e^{-10s}) - \dfrac{5}{s(s^2+1)}e^{-10s} \end{array}\right\} \cdots\cdots\cdots\cdots(1\text{pt})$$

$\because \dfrac{1}{s^2(s^2+1)} = \dfrac{1}{s^2} - \dfrac{1}{s^2+1}$ and $\dfrac{1}{s(s^2+1)} = \dfrac{1}{s} - \dfrac{s}{s^2+1}$ ⋯⋯⋯⋯(1pt for partial fractions)

$\therefore Y(s) = \dfrac{1}{s^2+1} + \left(\dfrac{1}{s^2} - \dfrac{1}{s^2+1}\right)(e^{-5s} - e^{-10s}) - 5\left(\dfrac{1}{s} - \dfrac{s}{s^2+1}\right)e^{-10s}$

$y(t) = \sin t + [(t-5) - \sin(t-5)]\mathscr{U}(t-5) - [(t-10) - \sin(t-10)]\mathscr{U}(t-10) - 5(1 - \cos(t-10))\mathscr{U}(t-10)$
$= \sin t + [t - 5 - \sin(t-5)]\mathscr{U}(t-5) - [t - 5 - \sin(t-10) - 5\cos(t-10)]\mathscr{U}(t-10)$
(1pt for $\mathscr{L}^{-1}\left\{\dfrac{1}{s^2+1}\right\} = \sin t$, 1pt for $\mathscr{L}^{-1}\left\{\dfrac{s}{s^2+1}\right\} = \cos t$)

6. (a) (8%) Find the orthogonal trajectories of the family of curves $y = C\tan x$, where C is an arbitrary constant.
   (b) (8%) Solve the differential equation $(\cos x)\cdot y' + (\sin x)\cdot y = \tan x$, $y(0) = 1$.

Solution:

(a) The standard steps to solve an orthogonal trajectories problem:
   (1) Find the slope of the tangent directions at each point in terms of x and y only.
   (2) Set up the differential equation that has the orthogonal trajectories as solution.
   (3) Solve the differential equation and express answer as orthogonal trajectories.
   (4) (optional) Check answer via sketch of curves or computing slopes of tangent directions.

$$y = C\tan x \quad \Rightarrow \quad \frac{dy}{dx} = C\sec^2 x$$

The expression needs to be depending on x and y, hence we use $C = y\cot x$.

$$\frac{dy}{dx} = y\cot x\sec^2 x = \frac{y}{\sin x\cos x}$$

Orthogonal

$$\frac{dy}{dx} = -\frac{\sin x\cos x}{y}$$

Solve via separable method

$$y\ dy = -\sin x\cos x\ dx \quad \Rightarrow \quad \frac{y^2}{2} = \frac{\cos^2 x}{2} + K = -\frac{\sin^2 x}{2} + K = \frac{1}{4}\cos(2x) + K$$

The answers can also be written as

$$y = \pm\sqrt{\cos^2 x + K} = \pm\sqrt{-\sin^2 x + K} = \pm\sqrt{\frac{\cos(2x) + K}{2}}$$

□

(b) The standard steps to solve a linear differential equation
   (1) Put in standard form $y' + P(x)y = Q(x)$.
   (2) Find the integrating factor $e^{\int P(x)\ dx}$.
   (3) Solve the differential equation using the integrating factor.
   (4) Use the given initial condition to find the constant value.
   (5) (optional) Check answer.

$$(\cos x)\cdot y' + (\sin x)\cdot y = \tan x \quad \Rightarrow \quad y' + (\tan x)\cdot y = \tan x\sec x$$

Integrating factor

$$e^{\int \tan x\ dx} = e^{\ln|\sec x|} = \sec x$$

$$((\sec x)\cdot y)' = \tan x\sec^2 x \quad \Rightarrow \quad (\sec x)\cdot y = \frac{\tan^2 x}{2} + C = \frac{\sec^2 x}{2} + C$$

Plug in $x = 0$ and $y = 1$ to get $C = 1$

$$y = \frac{\tan^2 x + 2}{2\sec x} = \frac{\sin^2 x + 2\cos^2 x}{2\cos x} = \frac{1 + \cos^2 x}{2\cos x} = \frac{1}{2}\sec x + \frac{1}{2}\cos x$$

□

6. Grading Scheme:

(a) Total of 8 points.

The main concepts to award points are (1) knowing to find $\frac{dy}{dx}$ (2) knowing how orthogonal slope is obtained (3) knowing how to solve a separable differential equation (4) knowing the solution is a family of curves (i.e., having a constant).

Rough outline: 2 points for finding $\frac{dy}{dx}$ in terms of x, y only. 2 points for setting up the new differential equation. 4 points for solving the differential equation.

Step-by-step: find $\frac{dy}{dx}$ (1pt), change C into x, y (1pt), negative reciprocal (2pts), separate (1pt), integrate both sides (2pts), get a constant (1pt).

Possible big mistake: failure to convert C into in terms of $x, y \Rightarrow$ maximum of 4 points.

(b) Total of 8 points.

The main concepts to award points are (1) knowing about linear differential equations (2) knowing how to compute the integrating factor (3) knowing to solve for the constant using the initial condition.

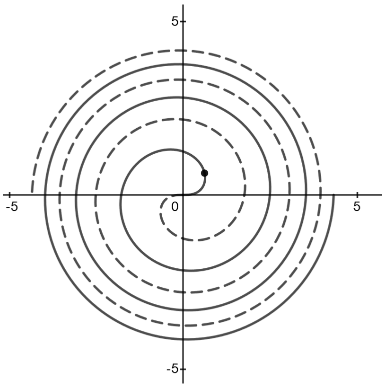
Rough outline: 3 points for finding the correct integrating factor. 3 points for solving the differential equation. 2 points for using the initial condition to solve the problem completely.

Step-by-step: divide by $\cos x$ (1pt), integrate $P(x)$ (1pt), obtain integrating factor (1pt), multiply by integrating factor and set up integral (1pt), integrating correctly (1pt), divide by integrating factor (1pt), find constant (1pt), state final answer (1pt).

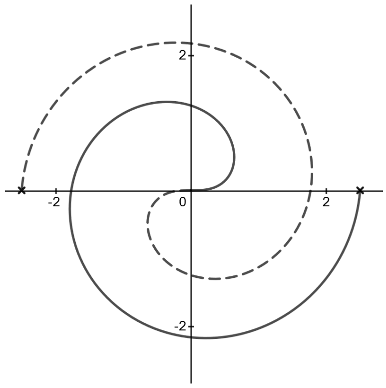
Possible big mistake: failure to put in standard form and get an integral that cannot be evaluated $\Rightarrow$ maximum of 4 points.

Remark: This grading scheme will be updated on exam day to reflect common mistakes found via browsing some exams.

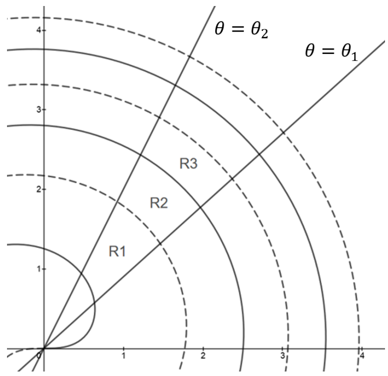
7. (14%) If you pour coffee mate into the coffee and stir it, the shape will be similar to a Fermat's spiral. The curve is given by $r^2 = \theta$, $\theta > 0$. Note that it can be realized as two curves $r = \sqrt{\theta}$(Solid curve) and $r = -\sqrt{\theta}$(Dotted curve).



Picture 1



Picture 2



Picture 3

(a) (4%) On the solid curve $r = \sqrt{\theta}$, compute the slope of the tangent line at $\theta = \dfrac{\pi}{4}$.

(b) (4%) Set up the integral, using the variable θ only, that expresses the length of the curve connecting $(x, y) = (\sqrt{2\pi}, 0)$ and $(x, y) = (-\sqrt{2\pi}, 0)$. **DO NOT** evaluate this integral!

(c) (6%) If we fix $0 < \theta_1 < \theta_2 < \dfrac{\pi}{2}$, among three regions R1, R2, R3(See Picture 3) cut by $\theta = \theta_1$, $\theta = \theta_2$ and the Fermat's spiral, which region has the largest area?

Solution:

(a) **(4pts)** $(x, y) = (r\cos\theta, r\sin\theta) = (\sqrt{\theta}\cos\theta, \sqrt{\theta}\sin\theta)$. Therefore,

$$\frac{dy}{dx}\Big|_{\theta=\frac{\pi}{4}} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}\Big|_{\theta=\frac{\pi}{4}} = \frac{\frac{1}{2\sqrt{\theta}}\sin\theta + \sqrt{\theta}\cos\theta}{\frac{1}{2\sqrt{\theta}}\cos\theta - \sqrt{\theta}\sin\theta}\Big|_{\theta=\frac{\pi}{4}} = \frac{2+\pi}{2-\pi}.$$

Partial credits:

- **IF** Write down the formula $\dfrac{dy}{dx} = \dfrac{r\cos\theta + \frac{dr}{d\theta}\sin\theta}{-r\sin\theta + \frac{dr}{d\theta}\cos\theta}$. **(3pts)**
- **ELSE IF** Express the curve as a parametric curve in terms of theta, $(x, y) = (\sqrt{\theta}\cos\theta, \sqrt{\theta}\sin\theta)$. **(2pts)**
- **ELSE IF** Write down $(x, y) = (r\cos\theta, r\sin\theta)$. **(1pt)**

(b) **(4pts)** The length is given by

$$L = 2\int_0^{2\pi} \sqrt{r^2 + (\frac{dr}{d\theta})^2}d\theta = 2\int_0^{2\pi} \sqrt{\theta + (\frac{1}{2\sqrt{\theta}})^2}d\theta = 2\int_0^{2\pi} \sqrt{\theta + \frac{1}{4\theta}}d\theta.$$

Partial credits for integrand:

- **IF** $ds = \sqrt{\theta + \dfrac{1}{4\theta}}d\theta$ **(2pts)**
- **ELSE IF** $ds = \sqrt{r^2 + (\dfrac{dr}{d\theta})^2}d\theta$ or $ds = \sqrt{(\dfrac{dx}{d\theta})^2 + (\dfrac{dy}{d\theta})^2}d\theta$. **(1pt)**

Partial credits for upper and lower bounds:

- **IF** Write $2\displaystyle\int_0^{2\pi}$, you will get **(2pts)**
- **ELSE IF** Separate the integration from 0 to 2π. **(1pt)**

(c) **(6pts)** The area formula is $dA = \dfrac{1}{2}r^2 d\theta$. The area **(2pts)** is given by

$$A = \int_{\theta_1}^{\theta_2} \frac{1}{2}\left(r_{\mathbf{out}}^2 - r_{\mathbf{in}}^2\right)d\theta = \frac{1}{2}\left(\int_{\theta_{\mathbf{out},1}}^{\theta_{\mathbf{out},2}} r^2 d\theta - \int_{\theta_{\mathbf{in},1}}^{\theta_{\mathbf{in},2}} r^2 d\theta\right).$$

Note: Writing $dA = \frac{1}{2}r^2 d\theta$ only worth **(1pt)**.

Therefore,

$$A_1 = \frac{1}{2}\int_{\theta_1}^{\theta_2}\left(\sqrt{\theta+\pi}^2 - \sqrt{\theta}^2\right)d\theta = \frac{1}{2}\left(\int_{\theta_1+\pi}^{\theta_2+\pi} r^2 d\theta - \int_{\theta_1}^{\theta_2} r^2 d\theta\right) = \frac{\pi}{2}(\theta_2-\theta_1)$$

$$A_2 = \frac{1}{2}\int_{\theta_1}^{\theta_2}\left(\sqrt{\theta+2\pi}^2 - \sqrt{\theta+\pi}^2\right)d\theta = \frac{1}{2}\left(\int_{\theta_1+2\pi}^{\theta_2+2\pi} r^2 d\theta - \int_{\theta_1+\pi}^{\theta_2+\pi} r^2 d\theta\right) = \frac{\pi}{2}(\theta_2-\theta_1)$$

$$A_3 = \frac{1}{2}\int_{\theta_1}^{\theta_2}\left(\sqrt{\theta+3\pi}^2 - \sqrt{\theta+2\pi}^2\right)d\theta = \frac{1}{2}\left(\int_{\theta_1+3\pi}^{\theta_2+3\pi} r^2 d\theta - \int_{\theta_1+2\pi}^{\theta_2+2\pi} r^2 d\theta\right) = \frac{\pi}{2}(\theta_2-\theta_1)$$

Credits:

- Use the correct equation to compute one of A_i's correct. **(3pts)**
- Compute all A_i's correct and state that they has the same area. **(1pt)**

Special case: If the answer is the region between 2 solid curves, the area should be $A_1 = \frac{1}{2}\int_{\theta_1}^{\theta_2}(\sqrt{\theta+2\pi}^2 - \sqrt{\theta}^2)d\theta = \pi(\theta_2-\theta_1)$. The student will get **(-1 pts)** for this but can still get the **(3pts)** and **(1pt)** listed above.

8. (14%) *The Apprentice* is a reality show on the *BBC* in which candidates need to compete in various business-related challenges. This week the candidates are required to create their own perfume. The following perfume is designed by one of them which consists of a stopper and a bottle (See Figure 1).

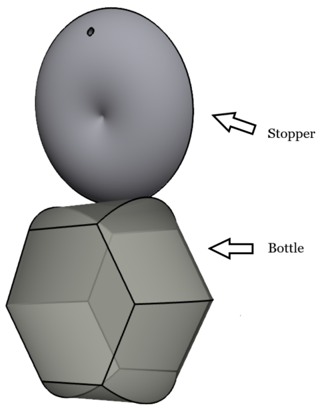


Figure 1
(The Perfume)

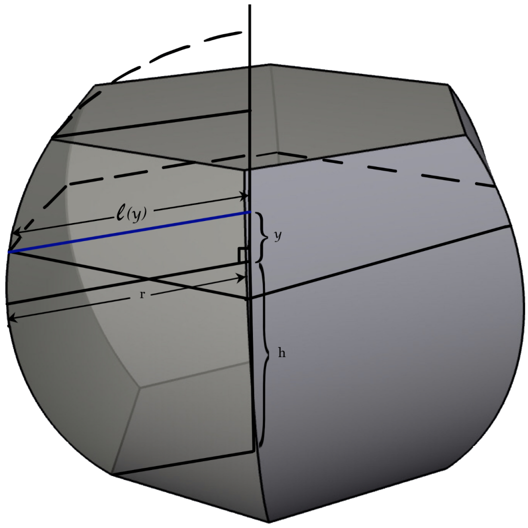


Figure 2 (Bottle)

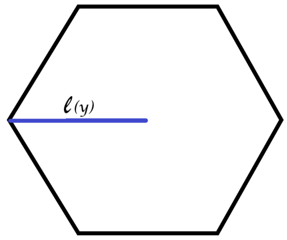


Figure 3
(Cross-section of the bottle at height y)

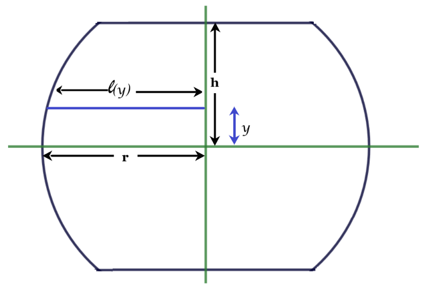


Figure 4
(Side-view of the bottle)

(a) (8%) Consider the region bounded by the curve $y = x\cos^{-1}(x)$ and the x-axis in the interval $0 \le x \le 1$. The stopper of the perfume is obtained by revolving this region about the y-axis. Find the volume of the stopper.

(b) (6%) The bottle of the perfume is designed such that

- the curved edges of the bottle are arcs of a circle of radius r (See Figure 2).
- each of its cross-section is a regular hexagon (see Figure 3),

The side-view of the bottle is given in Figure 4. Suppose the height of the bottle equals to $2h$ (with $h < r$).

i. Find the cross-sectional area of the bottle at a height y from its centre.
ii. Find the volume of the bottle.

Solution:

(a) By Shell method (or Pappus' Theorem), the required volume equals to

$$V = \int_0^1 2\pi x(x\cos^{-1}(x))dx \text{ (4M)}.$$

Partial credits zone :

- (at most 1M) Mentioning 'Shell method' or 'Pappus' Theorem', but without setting up any reasonable integrals
- (at most 2M) Setting up an integral with a correct integrand, up to a constant or sign, but with incorrect limits
- (at most 3M) Setting up an integral with a correct integrand with correct limits, but only up to a constant or sign

Using integration by parts,

$$V = \int_0^1 2\pi x(x\cos^{-1}(x))dx = 2\pi\left(\underbrace{\left[\frac{x^3\cos^{-1}(x)}{3}\right]_{x=0}^{x=1}}_{=0} + \frac{1}{3}\underbrace{\int_0^1 \frac{x^3}{\sqrt{1-x^2}}dx}_{=:I}\right).$$

To evaluate I, we put $x = \sin t$.

$$I = \int_0^{\frac{\pi}{2}} \sin^3(t)dt = \int_0^{\frac{\pi}{2}} (\cos^2(t) - 1)d\cos(t) = \left[\frac{\cos^3(t)}{3} - \cos(t)\right]_{t=0}^{t=\frac{\pi}{2}} = \frac{2}{3}.$$

Hence, the volume equals to $\frac{4\pi}{9}$.(4M)

Partial credits zone :

- A student getting no more than 1M above will receive 0M here.
- (at most 2M) A student who gets at least 2M above and also have a reasonable, yet incomplete computation.
- (at most 3M) A student who gets at least 2M above, evaluated the integral but made obvious mistakes in signs/scale factors.

(b) i. From Figure 4, we have $l(y) = \sqrt{r^2 - y^2}$ and hence

$$A(y) = 6 \cdot \frac{1}{2}(r^2 - y^2)\sin\frac{\pi}{3} = \frac{3\sqrt{3}}{2}(r^2 - y^2).(2M)$$

Partial credits zone : at most 1 marks in making any of the following mistakes

- Missing ‘6’ or missing ‘1/2’.
- Writing the exact value of $\sin\frac{\pi}{3}$ incorrectly.
- Forget to ‘square the square root’.

ii.

$$\underbrace{2\int_0^h A(y)dy}_{(2M)} = 3\sqrt{3}\int_0^h (r^2 - y^2)dy = \underbrace{3\sqrt{3}\left[r^2 y - \frac{y^3}{3}\right]_{y=0}^{y=h}}_{(1M)} = \underbrace{\sqrt{3}h(3r^2 - h^2)}_{(1M)}.$$

Partial credits zone : at most 1 marks in making any of the following mistakes

- Missing the 2 or having an incorrect limits of integration
- The second 1M will be awarded as long as the student integrated $r^2 - y^2$ correctly.