# 微乙小考六 (2019/6/13)

1. (4分) 計算 $\int_0^\infty x^{\frac{1}{2}} e^{-x} dx.$

sol: let $u = x^{\frac{1}{2}}, u^2 = x \Rightarrow dx = 2udu$

$$
\begin{aligned}
&\int_0^\infty ue^{-u^2} 2udu \\
=&\int_0^\infty 2u^2 e^{-u^2} du \\
=&2\int_0^\infty u^2 e^{-u^2} du \\
=&2[\int_0^\infty u(ue^{-u^2}) du] \\
=&2[-u(\frac{1}{2}e^{-u^2})|_0^\infty + \int_0^\infty \frac{1}{2}e^{-u^2} du] \\
=&2[\lim_{b\to\infty} \frac{1}{2}e^{-u^2}|_0^\infty + \int_0^b \frac{1}{2}e^{-u^2} du] = \frac{\sqrt{\pi}}{2}
\end{aligned}
$$

2. (4分) 計算 $\int_{-\infty}^\infty xe^{-x^2+x} dx.$

sol:

$$
\begin{aligned}
&\int_{-\infty}^\infty xe^{-x^2+x} dx \\
=&\int_{-\infty}^\infty xe^{-(x^2-x+\frac{1}{4}-\frac{1}{4})} dx \\
=&\int_{-\infty}^\infty xe^{-(x-\frac{1}{2})^2} e^{\frac{1}{4}} dx \\
=&e^{\frac{1}{4}} \int_{-\infty}^\infty ue^{-u^2} + \frac{1}{2}e^{-u^2} du \quad (let\ u = (\ x - \frac{1}{2})\ ) \\
=&e^{\frac{1}{4}}[0 + \frac{\sqrt{\pi}}{2}] \\
=&\frac{\sqrt{\pi}}{2} e^{\frac{1}{4}}
\end{aligned}
$$

3. (6分) 令 $f_X(t) = 3e^{-3t}$, $t \geq 0$. 求 $E(X)$ and $Var(X)$.

sol:

$$
\begin{aligned}
E[X] &= \int_{-\infty}^\infty t(3e^{-3t}) dt = \int_0^\infty t(3e^{-3t}) dt \\
&= 3[-\frac{1}{3}te^{-3t}|_0^\infty + \frac{1}{3}\int_0^\infty e^{-3t} dt] \\
&= 3[\lim_{b\to\infty} -\frac{1}{3}te^{-3t} - \frac{1}{9}e^{-3t}|_0^b] \\
&= 3[0 + \frac{1}{9}] = \frac{1}{3}
\end{aligned}
$$

$$
\begin{aligned}
E[X^2] &= \int_{-\infty}^{\infty} t^2(3e^{-3t})dt = \int_{0}^{\infty} t^2(3e^{-3t})dt \\
&= 3[-\frac{1}{3}t^2e^{-3t}|_0^\infty + \frac{2}{3}\int_0^\infty te^{-3t}dt] \\
&= 3[-\frac{1}{3}t^2e^{-3t}|_0^\infty + \frac{2}{3}[-\frac{1}{3}te^{-3t}|_0^\infty + \frac{1}{3}\int_0^\infty e^{-3t}dt]] \\
&= 3[\lim_{b\to\infty} -\frac{1}{3}t^2e^{-3t} - \frac{2}{9}te^{-3t} - \frac{2}{27}e^{-3t}|_0^b] \\
&= 3[0-(-\frac{2}{27})] = \frac{2}{9}
\end{aligned}
$$

$$
\begin{aligned}
Var(X) &= E[X^2] - (E[X])^2 \\
&= \frac{2}{9} - (\frac{1}{3})^2 = \frac{1}{9}
\end{aligned}
$$

4. (6分) 若 $f_X(t) = \frac{1}{\sqrt{\pi}}e^{-t^2}$ 與 $f_Y(t) = \frac{\sqrt{2}}{\sqrt{\pi}}e^{-2t^2}$. 且 $X$ 和 $Y$ 互相獨立。 令 $Z = X + Y$. 求 $f_Z(t)$.

sol:

$$
\begin{aligned}
f_Z(t) &= \int_{-\infty}^{\infty} f_X(t-s)f_Y(s)ds \\
&= \int_{-\infty}^{\infty} \frac{1}{\sqrt{\pi}}e^{-(t-s)^2}\frac{\sqrt{2}}{\sqrt{\pi}}e^{-2s^2}ds \\
&= \frac{\sqrt{2}}{\pi}\int_{-\infty}^{\infty} e^{-(t^2-2ts+s^2)-2s^2}ds \\
&= \frac{\sqrt{2}}{\pi}\int_{-\infty}^{\infty} e^{-3(s^2-\frac{2}{3}ts+\frac{t^2}{9}+\frac{2t^2}{9})}ds \\
&= \frac{\sqrt{2}}{\pi}e^{-\frac{2t^2}{3}}\int_{-\infty}^{\infty} e^{-3(s-\frac{t}{3})^2}ds(let\ x = \sqrt{3}(s-\frac{t}{3})) \\
&= \frac{\sqrt{2}}{\pi}e^{-\frac{2t^2}{3}}\int_{-\infty}^{\infty} e^{-x^2}\frac{1}{\sqrt{3}}dx \\
&= \frac{\sqrt{2}}{\pi}e^{-\frac{2t^2}{3}}\frac{\sqrt{\pi}}{\sqrt{3}} \\
&= \frac{\sqrt{2}}{\sqrt{3\pi}}e^{-\frac{2t^2}{3}}
\end{aligned}
$$