微乙小考三 (2019/04/11)

1. (5分) 試求 $g(x,y)=x^2+y^2$ 在 $2x^2+3xy+2y^2=1$ 限制條件下的最大值和最小值。

sol:

$$\begin{cases} \dfrac{\partial}{\partial x}\left(x^2+y^2\right) &= \lambda\dfrac{\partial}{\partial x}\left(2x^2+3xy+2y^2-1\right) \\ \dfrac{\partial}{\partial y}\left(x^2+y^2\right) &= \lambda\dfrac{\partial}{\partial y}\left(2x^2+3xy+2y^2-1\right) \\ 2x^2+3xy+2y^2 &= 1 \end{cases}$$

$$\Longrightarrow \begin{cases} 2x &= \lambda\left(4x+3y\right) \\ 2y &= \lambda\left(3x+4y\right) \\ 2x^2+3xy+2y^2 &= 1 \end{cases}$$

$$\Longrightarrow \begin{cases} x &= -1 \\ y &= 1 \\ \lambda &= 2 \end{cases} \text{or} \begin{cases} x &= 1 \\ y &= -1 \\ \lambda &= 2 \end{cases} \text{or} \begin{cases} x &= -\dfrac{\sqrt{7}}{7} \\ y &= -\dfrac{\sqrt{7}}{7} \\ \lambda &= \dfrac{2}{7} \end{cases} \text{or} \begin{cases} x &= \dfrac{\sqrt{7}}{7} \\ y &= \dfrac{\sqrt{7}}{7} \\ \lambda &= \dfrac{2}{7} \end{cases}$$

$$g\left(-1,1\right) = g\left(1,-1\right) = 2 \quad \text{(maximun)}$$

$$g\left(-\frac{\sqrt{7}}{7},-\frac{\sqrt{7}}{7}\right) = g\left(\frac{\sqrt{7}}{7},\frac{\sqrt{7}}{7}\right) = \frac{2}{7} \quad \text{(minimum)}$$

2. (5分) 求重積分 $\displaystyle\iint_{[0,1]\times[1,\sqrt{3}]}\frac{xdA}{(x^2+y^2)^2}$.

sol:

$$
\begin{aligned}
\iint_{[0,1]\times[1,\sqrt{3}]} \frac{x\,dA}{\left(x^2+y^2\right)^2} &= \int_1^{\sqrt{3}} \left(\int_0^1 \frac{x}{\left(x^2+y^2\right)^2}\,dx \right) dy \\
&= \int_1^{\sqrt{3}} \left(\left. \frac{-1}{2\left(x^2+y^2\right)} \right|_0^1 \right) dy \\
&= \int_1^{\sqrt{3}} \left(\frac{-1}{2\left(y^2+1\right)} + \frac{1}{2y^2} \right) dy \\
&= \left. \left(-\frac{1}{2}\tan^{-1} y - \frac{1}{2y} \right) \right|_1^{\sqrt{3}} \\
&= -\frac{\pi}{6} - \frac{\sqrt{3}}{6} + \frac{\pi}{8} + \frac{1}{2} \\
&= \frac{1}{2} - \frac{\sqrt{3}}{6} - \frac{\pi}{24}
\end{aligned}
$$

3. (5分) 求重積分 $\iint_\Omega \frac{\sin x dA}{xy}$，其中 $\Omega:\ \frac{\pi}{4} \le x \le \frac{\pi}{2},\ 1 \le y \le e^x.$

sol:

$$
\begin{aligned}
\iint_\Omega \frac{\sin x\,dA}{xy} &= \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \left(\int_1^{e^x} \frac{\sin x}{xy}\,dy \right) dx \\
&= \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \left(\left. \frac{\sin x \ln|y|}{x} \right|_1^{e^x} \right) dx \\
&= \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \sin x\,dx \\
&= -\cos y|_{\frac{\pi}{4}}^{\frac{\pi}{2}} \\
&= \frac{\sqrt{2}}{2}
\end{aligned}
$$

4. (5分) 求積分 $\int_0^4 \int_{y/2}^2 e^{x^2} dxdy.$

sol:

$$
\begin{aligned}
\int_0^4 \left(\int_{\frac{y}{2}}^2 e^{x^2}\,dx \right) dy &= \int_0^2 \left(\int_0^{2x} e^{x^2}\,dy \right) dx \\
&= \int_0^2 \left(\left. ye^{x^2} \right|_0^{2x} \right) dx \\
&= \int_0^2 2xe^{x^2}\,dx \\
&= \left. e^{x^2} \right|_0^2 \\
&= e^4 - 1
\end{aligned}
$$