微乙小考二 (2019/3/21)

1. (4 %) 令 $z = f(x, y)$, $x = ue^v \sin u$, $y = ue^v \cos u$，點 P 的 x, y 座標為 $(\frac{\pi e}{2}, 0)$, $\frac{\partial f}{\partial x}(P) = A$, $\frac{\partial f}{\partial y}(P) = B$。求在 $(u, v) = (\frac{\pi}{2}, 1)$ (i) $\frac{\partial z}{\partial u}(\frac{\pi}{2}, 1)$, (ii) $\frac{\partial z}{\partial v}(\frac{\pi}{2}, 1)$。(用 A, B 表之)

sol: (i)

$$\frac{\partial z}{\partial u}(\frac{\pi}{2}, 1) = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial u} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial u} = A(e \sin(\frac{\pi}{2}) + \frac{\pi}{2} e \cos(\frac{\pi}{2})) + B(e \cos(\frac{\pi}{2}) - \frac{\pi}{2} e \sin(\frac{\pi}{2}))$$

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(ii)

$$\frac{\partial z}{\partial u}(\frac{\pi}{2}, 1) = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial v} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial v} = A(\frac{\pi}{2} e \sin(\frac{\pi}{2})) + B(\frac{\pi}{2} e \cos(\frac{\pi}{2}))$$

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2. (6 %) $f(x, y) = x^2 y + e^{xy} \sin y$, $P = (1, 0)$

   (i) 求在 P 的梯度。

   (ii) 若 $\vec{u}$ ($|\vec{u}|$=1) 是 $f(x, y)$ 在 P 增加最快的方向。求 $\frac{\partial f}{\partial \vec{u}}(P)$.

sol: (i)

$$f_x = 2xy + ye^{xy} \sin(y) \Rightarrow f_x(1, 0) = 0$$

$$f_y = x^2 + xe^{xy} \sin(y) + e^{xy} \cos(y) \Rightarrow f_y(1, 0) = 2$$

The gradient is

$$\vec{f_P} = (0, 2)$$

(ii) The direction of the fastest increase must be the same direction as the gradient.

$$\vec{u} = \frac{\vec{f_P}}{|\vec{f_P}|} = (0, 1)$$

$$\Rightarrow \vec{f_P} \cdot \vec{u} = 2$$

3. (4 %) 求曲面 $S : 2z^3 - 3(x^2 + y^2)z + \tan^{-1}(xz) = c$ ($c =$ 某常數) 在點 $P = (1, 1, 1) \in S$ 的切平面方程式。

sol:

$$f_x = -6x + \frac{z}{1 + (xz)^2},\ f_y = -6y,\ f_z = 6z^2 - 3(x^2 + y^2) + \frac{x}{1 + (xz)^2}$$

代入(1,1,1)

$$f_x = -\frac{11}{2},\ f_y = -6,\ f_z = \frac{1}{2}$$

切平面方程式

$$(-\frac{11}{2}, -6, \frac{1}{2}) \cdot (x - 1, y - 1, z - 1) = 0$$

$$\Rightarrow 11x + 12y - 13z - 10 = 0$$

4. (6 %) 求 $f(x, y) = \frac{1}{x} + xy + \frac{1}{y}$ 的 (i)候選點 並 (ii) 決定其極值性質。

sol: (i)

$$f_x = \frac{1}{x^2} + y = 0$$

$$f_y = x + \frac{-1}{y^2} = 0$$

$\Rightarrow$

$$\begin{cases} y = \dfrac{1}{x^2} \\ x = \dfrac{1}{y^2} \end{cases} \Rightarrow (x, y) = (1, 1)$$

(ii)

$$D = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = \begin{vmatrix} \dfrac{2}{x^3} & -\dfrac{1}{x^2} \\ \dfrac{-1}{y^2} & \dfrac{2}{y^2} \end{vmatrix} = \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} = 3 > 0$$

And $f_{xx} = 2 > 0$ we can know (1,1) is minimal value