微乙小考一 (2019/3/7)

1. (9%) 求下列函數的偏導函數:

   (a) (2%) $f(x,y)=x^{\frac{1}{3}}y^2+\sin(xy)$. $\dfrac{\partial f}{\partial x}=?$ $\dfrac{\partial f}{\partial y}=?$

   (b) (4%) $f(x,y)=\displaystyle\int_{\sqrt{x}}^{xy}\sin(t^2)dt$. $\dfrac{\partial f}{\partial x}=?$ $\dfrac{\partial f}{\partial y}=?$

   (c) (3%) $f(x,y,z)=z^{\frac{x}{y}}$. $\dfrac{\partial f}{\partial x}=?$ $\dfrac{\partial f}{\partial y}=?$ $\dfrac{\partial f}{\partial z}=?$

sol: (a)
$$\frac{\partial f}{\partial x}=\frac{1}{3}x^{\frac{-2}{3}}y^2+y\cos(xy)$$
$$\frac{\partial f}{\partial y}=2x^{\frac{1}{3}}y+x\cos(xy)$$

(b)
$$\frac{\partial f}{\partial x}=\sin\left((xy)^2\right)\cdot\frac{\partial}{\partial x}(xy)-\sin\left(\sqrt{x}^2\right)\cdot\frac{\partial}{\partial x}\left(\sqrt{x}\right)=y\sin\left(x^2y^2\right)-\frac{1}{2}x^{\frac{-1}{2}}\sin x$$
$$\frac{\partial f}{\partial y}=\sin\left((xy)^2\right)\cdot\frac{\partial}{\partial y}(xy)-\sin\left(\sqrt{x}^2\right)\cdot\frac{\partial}{\partial y}\left(\sqrt{x}\right)=x\sin\left(x^2y^2\right)$$

(c)
$$\frac{\partial f}{\partial x}=z^{\frac{x}{y}}\cdot\ln z\cdot\frac{1}{y}$$
$$\frac{\partial f}{\partial y}=z^{\frac{x}{y}}\cdot\ln z\cdot\frac{-x}{y^2}$$
$$\frac{\partial f}{\partial x}=\frac{x}{y}\cdot z^{\frac{x}{y}-1}$$

2. (4%) 求 $f(x,y)=\tan^{-1}\left(\dfrac{y}{x}\right)$ 在點 $(1,1,\dfrac{\pi}{4})$ 的切面方程式。

sol:
$$\left.\frac{\partial f}{\partial x}\right|_{(1,1)}=\left.\frac{-yx^{-2}}{1+y^2x^{-2}}\right|_{(1,1)}=-\frac{1}{2}$$
$$\left.\frac{\partial f}{\partial y}\right|_{(1,1)}=\left.\frac{x^{-1}}{1+y^2x^{-2}}\right|_{(1,1)}=\frac{1}{2}$$

Hence the tangent plane we want is $z=\dfrac{\pi}{4}-\dfrac{1}{2}(x-1)+\dfrac{1}{2}(y-1)$.

3. (a) (4%) 若 $w=w(x,y)$ 且 $x=x(u,v)$, $y=y(u,v)$. 求 $\dfrac{\partial w}{\partial u}$, $\dfrac{\partial w}{\partial v}$.

   (b) (3%) 假設當 $(u,v)=(1,0)$ 時 $x(1,0)=1$, $y(1,0)=2$, 且 $\dfrac{\partial x}{\partial u}=-1$, $\dfrac{\partial x}{\partial v}=1$, $\dfrac{\partial y}{\partial u}=-2$, $\dfrac{\partial y}{\partial v}=1$. 當 $(x,y)=(1,2)$ 時 $\dfrac{\partial w}{\partial x}=1$, $\dfrac{\partial w}{\partial y}=2$. 用線性逼近估計 (u,v) 由 $(1,0)$ 改變到 $(0.9,0.01)$ 時 w 值的變化量。

sol: (a)
$$\frac{\partial w}{\partial u}=\frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial u}+\frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial u}$$
$$\frac{\partial w}{\partial v}=\frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial v}+\frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial v}$$

(b)
$$\left.\frac{\partial w}{\partial u}\right|_{(u,v)=(1,0)}=1\cdot(-1)+2\cdot(-2)=-5$$
$$\left.\frac{\partial w}{\partial v}\right|_{(u,v)=(1,0)}=1\cdot 1+2\cdot 1=3$$
$$\Delta w\approx(-5)\cdot\Delta u+3\cdot\Delta v=(-5)\cdot(-0.1)+3\cdot(0.01)=0.53$$