1072微乙01-05班期末考解答和評分標準

1. (20%) 解微分方程式

   (a) (10%) $y'(t) = (\tan t)(y^2 - 1)$, $y(0) = 2$. 求 $y(t)$，其中 t 在 0 的附近。

   (b) (10%) $t^2 y'(t) - 2ty(t) = 3t^{\frac{3}{2}}$, $y(1) = 0$. 求 $y(t)$，其中 $t > 0$.

Solution:

(a) 注意到$y(0) = 2$，$y(t)$的連續性保證在$t = 0$的附近，$y(t) > 1$，使用分離變數:

$$
\begin{aligned}
& y'(t) = (\tan t)(y^2(t) - 1) \\
\Rightarrow & \frac{y'(t)}{y^2(t) - 1} = \tan(t) && (y^2(t) - 1 \neq 0) \\
\Rightarrow & \int_0^t \frac{y'(s)}{y^2(s) - 1} \mathrm{d}s = \int_0^t \tan(s) \mathrm{d}s \\
\Rightarrow & \frac{1}{2} \int_0^t \frac{\mathrm{d}y(s)}{y(s) - 1} - \frac{1}{2} \int_0^t \frac{\mathrm{d}y(s)}{y(s) + 1} = [-\ln|\cos(s)|]_{s=0}^{s=t} \\
\Rightarrow & \frac{1}{2} \left[\ln \left| \frac{y(s) - 1}{y(s) + 1} \right| \right]_{s=0}^{s=t} = -\ln|\cos(t)| \\
\Rightarrow & \frac{1}{2} \ln \left| \frac{y(t) - 1}{y(t) + 1} \right| - \frac{1}{2} \ln \left| \frac{y(0) - 1}{y(0) + 1} \right| = -\ln|\cos(t)| \\
\Rightarrow & \frac{1}{2} \ln \left| \frac{y(t) - 1}{y(t) + 1} \right| - \frac{1}{2} \ln \left(\frac{1}{3} \right) = -\ln|\cos(t)| && (y(0) = 2) \\
\Rightarrow & \ln \left| \frac{y(t) - 1}{y(t) + 1} \right| = \ln \left(\frac{1}{3} \right) - 2\ln|\cos(t)| \\
\Rightarrow & \left| \frac{y(t) - 1}{y(t) + 1} \right| = \frac{1}{3\cos^2(t)} \\
\Rightarrow & \frac{y(t) - 1}{y(t) + 1} = \frac{1}{3\cos^2(t)} && (y(t) > 1) \\
\Rightarrow & y(t) = \frac{3\cos^2(t) + 1}{3\cos^2(t) - 1}
\end{aligned}
$$

配分:

- (3%)分離變數並積分
- (4%)積分正確
- (2%)初始條件代入正確
- (1%)最終答案

(b) 這是一階線性常微分方程，利用積分因子：

$$
\begin{aligned}
& t^2 y'(t) - 2ty(t) = 3t^{\frac{3}{2}} && (t \neq 0) \\
\Rightarrow & y'(t) - 2t^{-1}y(t) = 3t^{-\frac{1}{2}} \\
\Rightarrow & t^{-2} \cdot (y'(t) - 2t^{-1}y(t)) = t^{-2} \cdot 3t^{-\frac{1}{2}} && (\text{積分因子為} u(t) = \exp\left(\int -2t^{-1}\mathrm{d}t\right) = t^{-2}) \\
\Rightarrow & (t^{-2}y(t))' = 3t^{-\frac{5}{2}} \\
\Rightarrow & \int_1^t (s^{-2}y(s))'\mathrm{d}s = \int_1^t 3s^{-\frac{5}{2}}\mathrm{d}s \\
\Rightarrow & \left[\frac{y(s)}{s^2}\right]_{s=1}^{s=t} = \left[-2s^{-\frac{3}{3}}\right]_{s=1}^{s=t} \\
\Rightarrow & \frac{y(t)}{t^2} - \frac{y(1)}{1^2} = -2t^{-\frac{3}{2}} + 2 \cdot 1^{-\frac{3}{2}} \\
\Rightarrow & \frac{y(t)}{t^2} = -2t^{-\frac{3}{2}} + 2 && (y(1) = 0) \\
\Rightarrow & y(t) = -2\sqrt{t} + 2t^2
\end{aligned}
$$

配分：

- (3%)積分因子正確
- (4%)積分正確
- (2%)初始條件代入正確
- (1%)最終答案

2. (18%) 假設實驗室容器中的果蠅數目在第 t 天時為 $P(t)$，且 $P(t)$ 滿足微分方程式 $P'(t) = \lambda P(t)(M - P(t))$，其中 $\lambda, M > 0$，M 是果蠅數目的上限。

(a) (8%) 已知 $t = t_0 > 0$ 時，$P(t_0) = P_0$ 且 $0 < P_0 < M$. 求 $P(t)$。(請寫出解微分方程式的過程。)

(b) (4%) 若 $P(t) = \tilde{P}$ 時，果蠅數目增加率($P'(t)$)達到最大值，求 $\tilde{P}$(以 M 表示)。

(c) (6%) 研究人員發現當果蠅數達到 50 隻時，果蠅數目增加速率最大，並且 2 天後果蠅由 50 隻增加到 70 隻。估計再經過 2 天，牠們將由 70 隻增加到幾隻?(四捨五入到整數)

Solution:

(a)

$$P'(t) = \lambda P(t)\big(M - P(t)\big)$$

$$\frac{P'(t)}{P(t)\big(M - P(t)\big)} = \lambda$$

$$\frac{P'(t)}{M}\left(\frac{1}{P(t)} + \frac{1}{M - P(t)}\right) = \lambda$$

$$\frac{P'(t)}{1}\left(\frac{1}{P(t)} + \frac{1}{M - P(t)}\right) = \lambda M$$

$$\ln\left(\frac{P(t)}{M - P(t)}\right)\Big|_{P_0}^{P} = \lambda M(t - t_0)$$

$$\ln\left(\frac{P(t)}{M - P(t)}\right) - \ln\left(\frac{P_0(t)}{M - P_0(t)}\right) = \lambda M(t - t_0)$$

$$\frac{P(t)}{M - P(t)} = \mathrm{e}^{\lambda M(t-t_0)} * \left(\frac{P_0(t)}{M - P_0(t)}\right)$$

$$\frac{M - P(t)}{P(t)} = \frac{1}{\mathrm{e}^{\lambda M(t-t_0)} * \left(\frac{P_0(t)}{M - P_0(t)}\right)}$$

$$\frac{M}{P(t)} + 1 = \frac{1}{\mathrm{e}^{\lambda M(t-t_0)} * \left(\frac{P_0(t)}{M - P_0(t)}\right)}$$

$$P(t) = \frac{M}{\frac{1}{\mathrm{e}^{\lambda M(t-t_0)} * \left(\frac{P_0(t)}{M - P_0(t)}\right)} - 1}$$

什麼都沒寫+0

積分成功+6

寫出 I.C +8

(b)

$$P''^{(t)} = \big(\lambda P(t)M - \lambda P^2(t)\big)' = 0$$

$$\big(\lambda P(t)M - \lambda P^2(t)\big)' = \lambda P'^{(t)}M - 2P(t)\lambda * P'(t) = 0$$

$$P(t) = \frac{M}{2}$$

直接寫答案+3

有過程算錯+3

(c)

$$P(t) = \frac{M}{\frac{1}{e^{\lambda M(t-t_0)} * (\frac{P_0(t)}{M - P_0(t)})} - 1}$$

$$P(t_0) = \frac{M}{2}$$

$$P(t_0 + 2) = 70$$

$$P(t_0 + 4) = 84$$

有想出其他近似方法+4(例如$\frac{70}{50} = \cdots$，如果給到+3 再來要分~)

若不是這樣算但是算對+6

3. (10%) 假設 X 是一個隨機變數且

$$P_X(-1)=\frac{1}{5},\quad P_X(0)=\frac{1}{5},\quad P_X(1)=\frac{2}{5},\quad P_X(2)=\frac{1}{5}。$$

令 Y 是一個隨機變數滿足 $Y=X^2+3$。

(a) (4%) 求 Y 的可能取值範圍和它的機率函數。

(b) (6%) 求出 $\mathrm{E}(Y)$ 和 $\mathrm{Var}(Y)$。

Solution:

(a)

$$\begin{aligned} Y &= X^2+3 \Rightarrow Y=3,4,7(2pt)\\ f_Y(y=3) &= f_X(x=0)=\frac{1}{5}\\ f_Y(y=4) &= f_X(x=-1)+f_X(x=1)=\frac{3}{5}\\ f_Y(y=7) &= f_X(x=2)=\frac{1}{5}(2pt) \end{aligned}$$

(b)

$$\begin{aligned} E[Y] &= 3*\frac{1}{5}+4*\frac{3}{5}+7*\frac{1}{5}=\frac{3+12+7}{5}=4.4(2pt)\\ E[Y^2] &= 3^2*\frac{1}{5}+4^2*\frac{3}{5}+7^2*\frac{1}{5}=\frac{9+48+49}{5}=21.2(2pt)\\ Var(Y) &= E[Y^2]-(E[Y])^2=\frac{106}{5}-(\frac{22}{5})^2=1.84(2pt) \end{aligned}$$

4. (14%) 丟一個公正的銅板 100 次。 令 Z 是出現正面的次數。

(a) (6%) 求 $\mathrm{E}(Z)$ 和 $\mathrm{Var}(Z)$。

(b) (4%) 敘述 Chebyshev 不等式。

(c) (4%) 用 Chebyshev 不等式估計 $P(35 \le Z \le 65)$ 至少是多少。

Solution:

(a) Z為二項分配$B(100, \frac{1}{2})$。故

$$E(Z) = 100 \times \frac{1}{2} = 50,$$
$$\mathrm{Var}(Z) = 100 \times \frac{1}{2} \times (1 - \frac{1}{2}) = 25.$$

(b) 令X為一隨機變數，則Chebyshev不等式為

$$P\Big(|X - E(X)| \ge k\sqrt{\mathrm{Var}(X)}\Big) \le \frac{1}{k^2},$$

或可以寫成為

$$P\Big(|X - E(X)| \ge \varepsilon\Big) \le \frac{\mathrm{Var}(X)}{\varepsilon^2}.$$

(c) 由前兩小題可以知道

$$\begin{aligned} P(35 \le Z \le 65) &= P(|Z - 50| \le 15) \\ &= 1 - P(|Z - 50| > 15) \\ &\ge 1 - \frac{1}{9} \\ &= \frac{8}{9}. \end{aligned}$$

5. (10%) 設 X 的機率密度函數是 $f_X(t) = \frac{1}{\sqrt{\pi}} e^{-t^2}$。令 $Z = X^2$。

(a) (4%) 寫下 Z 的分配函數(不需要算出積分)。

(b) (6%) 求 Z 的機率密度函數。

Solution:

(a) CDF (Cumulative Distribution Function)= $F_z(t)$

$$F_z(t) = P(Z \le t) = P(X^2 \le t) = P(-\sqrt{t} \le X \le \sqrt{t}) = \int_{-\sqrt{t}}^{\sqrt{t}} f_x(s)ds = \int_{-\sqrt{t}}^{\sqrt{t}} \frac{1}{\sqrt{\pi}} e^{-s^2} ds$$

note 1: you better use dummy variable for the integration. If you didn't do it this time, you won't lose any point.

note 2: If you didn't write the process precisely, you will only get 2 points at most.

(b) PDF (Probability density function)= $f_z(t)$

$$f_z(t) = \frac{dF_z}{dt} = \frac{d(\int_{-\sqrt{t}}^{\sqrt{t}} \frac{1}{\sqrt{\pi}} e^{-s^2} ds)}{dt} = 2 \times \frac{d(\int_{0}^{\sqrt{t}} \frac{1}{\sqrt{\pi}} e^{-s^2} ds)}{dt} = 2 \times \frac{1}{\sqrt{\pi}} e^{-\sqrt{t}^2} \times (\sqrt{t})'$$

(by fundamental thm of calculus)

$$= 2 \times \frac{1}{\sqrt{\pi}} e^{-t} \times \frac{1}{2} \times t^{\frac{-1}{2}} = \frac{1}{\sqrt{\pi}} \times e^{-t} \times t^{\frac{-1}{2}}, t > 0$$

$$0, t \le 0$$

note 1: you would get 3 points if know the pdf is the derivative of cdf.

note 2: If you lose 1 point if you didn't write the domain of the cdf.

6. (12%)

(a) (5%) 若 $P_X(i) = \frac{m^i}{i!}e^{-m}$, $P_Y(j) = \frac{n^j}{j!}e^{-n}$, $i, j = 0, 1, 2, \cdots$, $m, n > 0$, 且 X, Y 獨立。令 $Z = X + Y$，證明 $P_Z(k) = \frac{(m+n)^k}{k!}e^{-(m+n)}$.

(b) (7%) 已知早上和下午尖峰時間的車禍件數各自呈 Poisson 分配，平均每個早上發生 1 件，每個下午發生 2 件。若兩時段發生車禍數互相獨立，求一天內兩尖峰時段總計發生 3 件以上車禍的機率。

Solution:

(a)

$$P_Z(k) = P(Z = k) = P(X + Y = k) = \sum_{i=0}^{k} P(X = i, Y = k - i) \qquad (1)$$

Since X and Y are independent, we have

$$\begin{aligned}
\sum_{i=0}^{k} P(X = i, Y = k - i) &= \sum_{i=0}^{k} P(X = i)P(Y = k - i) \qquad (2) \\
&= \sum_{i=0}^{k} \Big(\frac{m^i}{i!}e^{-m}\Big) \cdot \Big(\frac{n^{k-i}}{(k-i)!}e^{-n}\Big) \\
&= e^{-(m+n)} \sum_{i=0}^{k} \frac{1}{i!(k-i)!} m^i \cdot n^{k-i} \\
&= \frac{e^{-(m+n)}}{k!} \sum_{i=0}^{k} \frac{k!}{i!(k-i)!} m^i \cdot n^{k-i} \\
&= \frac{e^{-(m+n)}}{k!} \sum_{i=0}^{k} C_i^k m^i \cdot n^{k-i} \\
&= \frac{e^{-(m+n)}}{k!} (m+n)^k
\end{aligned}$$

(b) The total number of accident in morning and evening is still a Poisson distribution, and its means is $1 + 2 = 3$.

Hence its probability function is $P(k) = \frac{3^k}{k!}e^{-3}$

The probability of no accident in morning and evening is $\frac{3^0}{0!}e^{-3} = e^{-3}$

The probability of one accident in morining and evening is $\frac{3^1}{1!}e^{-3} = 3e^{-3}$

The probability of two accident in morining and evening is $\frac{3^2}{2!}e^{-3} = \frac{9}{2}e^{-3}$

Therefore, the probability of having 3 or more accidents in morning and evening is $1 - e^{-3} - 3e^{-3} - \frac{9}{2}e^{-3} = 1 - \frac{17}{2}e^{-3}$

Grading Criteria

(a) Write down (1) get 1 point

Use the independent property and write down (2) get 2 points

The rest of 2 points depend on your proof, you should explain each step you wrote down.

Note that Poisson distribution is a discrete probability distribution, you should use summation instead of integration!

(b) 2 points for knowing the number of accident in morning and evening is still poisson distribution and its means is 3

Calculate the probability of 0, 1, 2 accident in morning and evening correctly get 1 point respectively

The rest of 2 point depend on your calculation of final answer.

7. (16%) 假設收銀員甲、乙服務一位客人的時間分別呈指數分配，平均時間各為 2 分鐘和 4 分鐘，並且甲、乙的服務是相互獨立的。

(a) (6%) 甲正開始服務一位客人，有另一人排隊。此時你排甲的櫃檯，等候時間的機率密度函數是 $f_X(t) = \frac{1}{4}te^{-\frac{1}{2}t}$, $t > 0$. 求等候時間的期望值。

(b) (10%) 承上題，乙正開始服務一位客人，沒有人排隊。求你由甲櫃檯換到乙櫃檯會提早接受到服務的機率。

Solution:

(a)

$$
\begin{aligned}
E(X) &= \int_0^\infty t \cdot \frac{1}{4}te^{-\frac{t}{2}}\,dt && (2\%) \\
&= \left(-\frac{1}{2}t^2e^{-\frac{t}{2}}\right)\Big|_0^\infty + \int_0^\infty te^{-\frac{t}{2}}\,dt && (1\%) \\
&= \left(-\frac{1}{2}t^2e^{-\frac{t}{2}}\right)\Big|_0^\infty + \left(-2te^{-\frac{t}{2}}\right)\Big|_0^\infty + \int_0^\infty 2e^{-\frac{t}{2}}\,dt && (1\%) \\
&= \left(-\frac{1}{2}t^2e^{-\frac{t}{2}}\right)\Big|_0^\infty + \left(-2te^{-\frac{t}{2}}\right)\Big|_0^\infty + \left(-4e^{-\frac{t}{2}}\right)\Big|_0^\infty \\
&= 0 + 0 + 4 \\
&= 4 && (2\%)
\end{aligned}
$$

(b)

(2%)排乙時等候時間Y，則

$$f_Y(y) = \frac{1}{4}e^{-\frac{y}{4}}$$

(2%)聯合機率密度

$$f(x,y) = f_X(x) \cdot f_Y(y) = \frac{1}{16}xe^{-\frac{x}{2}}e^{-\frac{y}{4}}$$

(6%)提早接受到服務的機率

$$
\begin{aligned}
P &= \iint_{0 \le y \le x} f(x,y)\,dx\,dy \\
&= \int_0^\infty \int_0^x \frac{1}{16}xe^{-\frac{x}{2}}e^{-\frac{y}{4}}\,dy\,dx \\
&= \int_0^\infty \left(-\frac{1}{4}xe^{-\frac{3x}{4}} + \frac{1}{4}xe^{-\frac{x}{2}}\right)dx \\
&= \frac{5}{9}
\end{aligned}
$$