1. (11 pts) Railway tracks cannot have large curvature at any point, otherwise the train might derail. Let $\mathbf{r}(t) = \langle 20t, a(t-t^2), 5\rangle$, $a \neq 0$ be the vector function describing a track starting at $(0,0,5)$ and ending at $(20,0,5)$.
   (a) (5 pts) Find the unit tangent vector $\vec{T}(t)$ and the unit normal vector $\vec{N}(t)$.
   (b) (6 pts) If the curvature needs to be smaller than 0.001, what is the largest possible value of a?

Solution:

(a) Solution 1: Book method.

By definition

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}$$

Here

$$\mathbf{r}'(t) = \langle 20, a(1-2t), 0\rangle, \quad |\mathbf{r}'(t)| = \sqrt{400 + a^2(1-2t)^2}$$

Therefore

$$\mathbf{T}(t) = \left\langle \frac{20}{\sqrt{400+a^2(1-2t)^2}}, \frac{a(1-2t)}{\sqrt{400+a^2(1-2t)^2}}, 0\right\rangle$$

Next is to find $\mathbf{N}(t)$. By definition

$$\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|}$$

Here

$$\mathbf{T}'(t) = \left\langle \frac{40a^2(1-2t)}{(400+a^2(1-2t)^2)^{3/2}}, \frac{-2a}{\sqrt{400+a^2(1-2t)^2}} + \frac{2a^3(1-2t)^2}{(400+a^2(1-2t)^2)^{3/2}}, 0\right\rangle$$

$$\mathbf{T}'(t) = \left\langle \frac{40a^2(1-2t)}{(400+a^2(1-2t)^2)^{3/2}}, \frac{-800a}{(400+a^2(1-2t)^2)^{3/2}}, 0\right\rangle = \frac{40a}{(400+a^2(1-2t)^2)^{3/2}}\langle a(1-2t), -20, 0\rangle$$

$$|\mathbf{T}'(t)| = \frac{40|a|}{400+a^2(1-2t)^2}$$

Solution 2: **TNB** method.

Find the first and second derivative of the vector function

$$\mathbf{r}'(t) = \langle 20, a(1-2t), 0\rangle$$

$$\mathbf{r}''(t) = \langle 0, -2a, 0\rangle$$

The cross product $\mathbf{r}'(t) \times \mathbf{r}''(t)$ is in the same direction as $\mathbf{B}(t)$.

$$\mathbf{r}'(t) \times \mathbf{r}''(t) = \langle 0, 0, -40a\rangle$$

and we know $\mathbf{N}(t)$ is in the direction of $\mathbf{B}(t) \times \mathbf{T}(t)$ when $a \neq 0$

$$\langle 0, 0, -40a\rangle \times \langle 20, a(1-2t), 0\rangle = \langle 40a^2(1-2t), -800a, 0\rangle$$

Make the vectors unit length

$$\mathbf{T}(t) = \left\langle \frac{20}{\sqrt{400 + a^2(1-2t)^2}}, \frac{a(1-2t)}{\sqrt{400 + a^2(1-2t)^2}}, 0 \right\rangle$$

$$\mathbf{N}(t) = \left\langle \frac{a(1-2t)}{\sqrt{400 + a^2(1-2t)^2}} \left(\frac{a}{|a|}\right), \frac{-20}{\sqrt{400 + a^2(1-2t)^2}} \left(\frac{a}{|a|}\right), 0 \right\rangle$$

Other possible solutions: Use geometry to find $\mathbf{N}(t)$. Since the space curve is on a plane we can use the slope of $\mathbf{T}(t)$ to find the slope of $\mathbf{N}(t)$. Orthogonal projection of $\mathbf{r}''(t)$ to $\mathbf{r}'(t)$ can also find $\mathbf{N}(t)$.

(For all methods) Grading scheme: 2 points for $\mathbf{T}(t)$ and 2 points for $\mathbf{N}(t)$ in $a > 0$ case (for each vector 1 point for direction 1 point for unit length). 1 point for correctly identifying the cases.

For example: 2 points if the answers are not unit vectors and only for $a > 0$.

(b) Solution 1: Use answer in (a) and definition.

By definition of the curvature $\kappa(t)$

$$\kappa(t) = \frac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|}$$

Use answer in (a) to get

$$\kappa(t) = \frac{40|a|}{(400 + a^2(1-2t)^2)^{3/2}}$$

The curvature is largest when $t = \frac{1}{2}$ (when the denominator is minimized). So we can set $\kappa(\frac{1}{2}) < 0.001$.

$$\frac{40|a|}{8000} < 0.001, \quad |a| < 0.2.$$

The largest possible value of a is 0.2.

Solution 2: Use formula in book.

$$\kappa(t) = \frac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3}$$

$$\kappa(t) = \frac{40|a|}{(400 + a^2(1-2t)^2)^{3/2}}$$

And the rest of the problem is the same.

(For all methods) Grading scheme: 1 point for a formula for $\kappa(t)$, 3 points for finding the correct expression (absolute value is 1 point), 2 points for analyzing and getting the answer (1 point for realizing $t = \frac{1}{2}$, 1 point solving inequality).

2. (12 pts) Let $f(x,y) = \begin{cases} \sin \dfrac{xy^2}{x^2+y^4}, & \text{if } (x,y) \neq (0,0) \\ 0, & \text{if } (x,y) = (0,0) \end{cases}$.

(a) (6 pts) Compute f_x and f_y for all (x,y) including $(0,0)$.

(b) (6 pts) Is $f(x,y)$ continuous at $(0,0)$? Is $f(x,y)$ differentiable at $(0,0)$? Justify your answers.

Solution:

(a) for $(x,y) \neq (0,0)$

$$f_x = \left(\cos \frac{xy^2}{x^2+y^4}\right)\left(\frac{y^2(x^2+y^4) - xy^2 \cdot 2x}{(x^2+y^4)^2}\right) = \left(\cos \frac{xy^2}{x^2+y^4}\right)\frac{y^6 - x^2y^2}{(x^2+y^4)^2} \qquad \text{(2 points)}$$

$$f_y = \left(\cos \frac{xy^2}{x^2+y^4}\right)\left(\frac{2xy(x^2+y^4) - xy^2(4y^3)}{(x^2+y^4)^2}\right) = \left(\cos \frac{xy^2}{x^2+y^4}\right)\frac{2x^3y - 2xy^5}{(x^2+y^4)^2} \qquad \text{(2 points)}$$

$$f_x(0,0) = \lim_{x\to 0} \frac{f(x,0) - f(0,0)}{x} = \lim_{x\to 0} \frac{0}{x} = 0 \qquad \text{(1 point)}$$

$$f_y(0,0) = \lim_{y\to 0} \frac{f(0,y) - f(0,0)}{y} = 0 \qquad \text{(1 point)}$$

(b) If $f(x,y)$ is continuous at $(0,0)$, by definition any path $(x(t), y(t)) \to (0,0)$. We have $\lim_{t\to 0} f(x(t), y(t)) = f(0,0)$

$$\text{along } x = y^2,\ \lim_{y\to 0} f(y^2, y) = \lim_{y\to 0} \sin \frac{y^4}{y^4+y^4} = \sin \frac{1}{2} \qquad \text{(2 points)}$$

$$\sin \frac{1}{2} \neq f(0,0).\ \therefore f(x,y) \text{ is not continuous at } (0,0) \qquad \text{(1 point)}$$

or

Take two paths with different limits (2 points)

different limits $\Rightarrow \lim_{(x,y)\to(0,0)} f(x,y)$ does not exist $\therefore$ f is not continuous at $(0,0)$

(1 point)

If $f(x,y)$ is differentiable at $(0,0)$, it means that $f(x,y) = f(0,0) + f_x(0,0)x + f_y(0,0)y + \epsilon_1 x + \epsilon_2 y$

$\lim_{(x,y)\to(0,0)} \epsilon_1 = 0$, $\lim_{(x,y)\to(0,0)} \epsilon_2 = 0$ (the definition of differentiability 1 point)

$\therefore \lim_{(x,y)\to(0,0)} f(x,y) = f(0,0) \Rightarrow f(x,y)$ is continuous at $(0,0)$ (1 point)

Because $f(x,y)$ is not continuous at $(0,0)$, it implies that $f(x,y)$ is not differentiable at $(0,0)$ (1 point)

Remark: If give wrong reasons or not state the reason for not differentiability, clearly only get one point.

3. (12 pts) Consider the level surface $z^2 + z\tan^{-1}\dfrac{y}{x} = \dfrac{\pi}{4} + 1$.

(a) (5 pts) Find the tangent plane for the level surface at $(1,1,1)$.

(b) (4 pts) The level surface defines z implicitly as a function of x and y, $z = f(x,y)$. Compute the directional derivative of $f(x,y)$ at $(1,1)$ in the direction of $-3\vec{i} + 4\vec{j}$.

(c) (3 pts) Use linear approximation to estimate the value of $f(0.98, 1.04)$.

Solution:

(a) Define $F(x,y,z) = z^2 + z\tan^{-1}\dfrac{y}{x}$

$$\nabla F = \left(z\frac{-\frac{y}{x^2}}{1+\left(\frac{y}{x}\right)^2}, z\frac{\frac{1}{x}}{1+\left(\frac{y}{x}\right)^2}, 2z + \tan^{-1}\frac{y}{x} \right) = \left(\frac{-yz}{x^2+y^2}, \frac{xz}{x^2+y^2}, 2z + \tan^{-1}\frac{y}{x} \right) \qquad \text{(2 points)}$$

$$\nabla F(1,1,1) = (-\frac{1}{2}, \frac{1}{2}, 2 + \tan^{-1}1) = (-\frac{1}{2}, \frac{1}{2}, 2 + \frac{\pi}{4}) \qquad \text{(1 point)}$$

tangent plane $(-\frac{1}{2})(x-1) + \frac{1}{2}(y-1) + (2+\frac{\pi}{4})(z-1) = 0$

$$\Rightarrow -\frac{x}{2} + \frac{y}{2} + (2+\frac{\pi}{4})z - (2+\frac{\pi}{4}) = 0$$

$$\Rightarrow x - y - (4+\frac{\pi}{2})z + (4+\frac{\pi}{2}) = 0 \qquad \text{(2 points)}$$

The problem (b) and (c) need to specify at $(1,1,1)$. Please see remark below.

(b) $F(x,y,z(x,y)) = \dfrac{\pi}{4} + 1 \Rightarrow \dfrac{\partial F}{\partial x} + \dfrac{\partial F}{\partial z}\dfrac{\partial z}{\partial x} = 0$

$$\because \left.\frac{\partial F}{\partial z}\right|_{(1,1,1)} \neq 0$$

$$\therefore \left.\frac{\partial z}{\partial x}\right|_{(1,1,1)} = -\left.\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial z}}\right|_{(1,1,1)} = -\frac{-\frac{1}{2}}{(2+\frac{\pi}{4})} = \frac{1}{4+\frac{\pi}{2}} \qquad \text{(1 point)}$$

Similarly, $\left.\dfrac{\partial z}{\partial y}\right|_{(1,1,1)} = -\left.\dfrac{\frac{\partial F}{\partial y}}{\frac{\partial F}{\partial z}}\right|_{(1,1,1)} = -\dfrac{1}{4+\frac{\pi}{2}}$ (1 point)

$u = (-\dfrac{3}{5}, \dfrac{4}{5})$, $D_u f = <\nabla f, u> = \dfrac{-3}{5(4+\frac{\pi}{2})} - \dfrac{4}{5(4+\frac{\pi}{2})} = \dfrac{-7}{5(4+\frac{\pi}{2})}$ (2 points)

(c)

$$f(0.98, 1.04) \approx 1 + \frac{1}{4+\frac{\pi}{2}}(0.98-1) - \frac{1}{4+\frac{\pi}{2}}(1.04-1) \qquad \text{(2 points)}$$

$$\approx 1 - \frac{0.02+0.04}{4+\frac{\pi}{2}} = 1 - \frac{0.06}{4+\frac{\pi}{2}} \qquad \text{(1 point)}$$

Remark:

for $x = 1$, $y = 1$, $z^2 + z\tan^{-1}1 = \dfrac{\pi}{4} + 1$

That is, $z^2 + \dfrac{\pi}{4}z = \dfrac{\pi}{4} + 1 \Rightarrow z = 1$ or $z = -1 - \dfrac{\pi}{4}$

For $z = 1$, as discussed above.

For $z = -1 - \dfrac{\pi}{4}$, $\nabla F(1,1,-1-\dfrac{\pi}{4}) = \left(\dfrac{1+\frac{\pi}{4}}{2}, \dfrac{-1-\frac{\pi}{4}}{2}, -2-\dfrac{\pi}{4} \right)$

4. (12 pts) Find and classify all critical points of $f(x,y) = 4x^3 + 2xy^2 + \frac{2}{3}y^3 + 6x^2$.

Reminder: each critical point must be shown to be either a local maximal point, a local minimal point, or neither of the above.

Solution:

Step 1: Compute the gradient (partial derivatives) **3pts**.

We have $f_x = 12x^2 + 12x + 2y^2$, $f_y = 4xy + 2y^2 = 2y(2x + y)$.

Partial credits: If the student only get 1 of them correct, 2pts.

Step 2: Find the critical points. **3pts**

Set $f_x = f_y = 0$, we can find the critical points. From $f_y = 0$, either $y = 0$ or $y = -2x$. If $y = 0$, from $f_x = 12x^2 + 12x - 2y^2 = 0$, we have $(0,0)$ or $(-1,0)$. If $y = -2x$, from $f_x = 12x^2 + 12x + 2y^2 = 12x^2 + 12x + 8x^2 = 0$, we have $(0,0)$ or $(-\frac{3}{5}, \frac{6}{5})$.

Partial credits for this part:

- Only set $f_x = f_y = 0$ without fining any correct critical points. 1pts
- If the student find some correct critical points but not all the critical points or there is some incorrect critical points. 2pts

Step 3: Second derivative test. **6pts**

Now, consider the second derivative test. $f_{xx} = 24x + 12$, $f_{xy} = f_{yx} = 4y$, $f_{yy} = 4x + 4y$. $D = 48(2x + 1)(y + x) - 16y^2 = 16(-y^2 + (6x + 3)(y + x))$.

- At $(-1,0)$, we have $D = 48 > 0$, $f_{xx} = -12 < 0$, it is a local maximum.
- At $(-\frac{3}{5}, \frac{6}{5})$, we have $D = 16(-(\frac{6}{5})^2 - \frac{3}{5} \cdot \frac{3}{5}) < 0$, it is neither a local maximum nor a local minimum.
- At $(0,0)$, $D = 0$, the second derivative test fails. However, if we approach $(0,0)$ from y axis, $f(0,y) = \frac{2}{3}y^3$. $f(0,y) > 0$ for positive y, $f(0,y) < 0$ for negative y. Therefore, $(0,0)$ is neither a local maximum nor a local minimum.

Partial credits for this part:

- 2pts for each correct pair of critical point and the classification. it is okay to write "Saddle" in the case of "Neither a local max nor a local min".
- For $(0,0)$,
  - If the answer only says "Second derivative fails" 1pts.
  - If the answer is "Neither a local max nor a local min" or "Saddle" 2pts.
- If there is no correct pair of critical point and the classification, but the student knows how to use second derivative test, the student can obtain at most 3pts for this part.

5. (15 pts) Consider the part of an elliptic paraboloid defined by $z = \frac{x^2}{16} + \frac{y^2}{8}$, $z \leq 6$. Find the points on the surface segment which are respectively the farthest from and the closest to the point $(0, 0, 8)$.

Solution:

Step 1 Compute the interior candidate for extremal value. **7pts**

Let $f(x, y, z) = x^2 + y^2 + (z - 8)^2$. Let $g(x, y, z) = \frac{x^2}{16} + \frac{y^2}{8} - z = 0$ be the constraint. In the interior of the surface, the extremal point happens when $\nabla f = (2x, 2y, 2(z - 8))$ is parallel to $\nabla g = (\frac{x}{8}, \frac{y}{4}, -1)$. From the x, y components, we have $\frac{xy}{2} = \frac{xy}{4}$. Either $x = 0$ or $y = 0$.

- If $(x, y) = (0, 0)$, we have $z = 0$ and $f(0, 0, 0) = 64$.
- If $x = 0$, $y \neq 0$, we have $z = 8 - 4 = 4$, $y^2 = 32$, $y = \pm 4\sqrt{2}$, $f(x, y, z) = 32 + 16 = 48$.
- If $y = 0$, $x \neq 0$, we have $z = 0$. We have $x = 0$ again.

Partial credits for this part:

- Setting up the function f and the constraint $g = 0$. 1pt
- Correct answer for ∇f, ∇g. worth 1pt each.
- Setting up the equation $\nabla f = \lambda \nabla g$. 1pt.
- Solving $\nabla f = \lambda \nabla g$ together with the constraint. 3pts. Each solution worth 1pt.
- It is possible to solve this problem without using Lagrange multiplier, partial credits may be given to incomplete answers.

Step 2 Compute the boundary candidate for extremal value. **6pts**

Let $h(x, y, z) = z - 6$. At the intersection of g and h, we consider it as a optimization problem with 2 constraint. From $\nabla h = (0, 0, 1)$, use Lagrange multiplier, we have

$$\nabla f = \lambda \nabla g + \mu \nabla h.$$

We obtain the equation $(2x, 2y, 2(z - 8)) = (\lambda \frac{x}{8}, \lambda \frac{y}{4}, -\lambda + \mu)$ together with 2 constraints $g = 0$, $h = 0$. From $h = 0$, we have $z = 6$. $(2x, 2y, -4) = (\lambda \frac{x}{8}, \lambda \frac{y}{4}, -\lambda + \mu)$.

Use the same argument as above, we have either $x = 0$ or $y = 0$.

- If $(x, y) = (0, 0)$, we have $z = 0$. This contradicts $z = 6$
- If $x = 0$, $z = 6$, we have $y^2 = 48$, $y = \pm 4\sqrt{3}$, $f(x, y, z) = 48 + 4 = 52$.
- If $y = 0$, $z = 6$, we have $x^2 = 96$, $x = \pm 4\sqrt{6}$, $f(x, y, z) = 96 + 4 = 100$.

Partial credits for this part:

- Setting up the constraint $h = 0$ or other ways to describe the boundary. 1pt

- Setting up the equation $\nabla f = \lambda \nabla g + \mu \nabla h$. 2pts.
- Solving $\nabla f = \lambda \nabla g + \mu \nabla h$ together with the constraint. 3pts.
- It is possible to solve this problem without using Lagrange multiplier(Eg. paramatrize the boundary ellipse), partial credits may be given to incomplete answers.

Step 3 Conclusion. **2pts**

- The longest distance happens at $(\pm\sqrt{96}, 0, 6)$ and the distance is 10.(1pt)
- The shortest distance happens at $(0, \pm 4\sqrt{2}, 4)$ and the distance is $4\sqrt{3}$.(1pt)

6. (12 pts) (a) (5 pts) Evaluate the iterated integral $\int_0^8 \int_{x^{\frac{1}{3}}}^2 \frac{1}{1+y^4}\, dy\, dx$.

(b) (7 pts) Evaluate the double integral $\iint_{\mathcal{R}} (x^2+y^2)\, dA$, where $\mathcal{R}$ is the region bounded by the ellipse $9x^2+y^2=36$.

Solution:

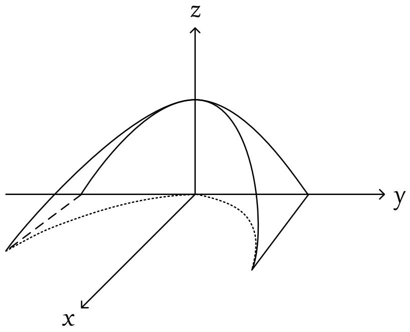
(a)

$$\begin{aligned} I &= \int_0^8 \int_{x^{\frac{1}{3}}}^2 \frac{1}{1+y^4}\, dy\, dx = \int_{y=0}^2 \int_{x=0}^{y^3} \frac{1}{1+y^4}\, dx\, dy && \text{(到此步驟得 3分)} \\ &= \int_{y=0}^2 \frac{y^3}{1+y^4}\, dy && \text{(到此步驟得 4分)} \\ &= \left[\frac{1}{4}\ln\left|1+y^4\right|\right]_0^2 && \text{(到此步驟得 5分)} \\ &= \frac{1}{4}\ln 17. \end{aligned}$$

(b)

$$\begin{aligned} I &= \int\int_{9x^2+y^2=36} (x^2+y^2)\, dA \quad (\text{let } x=2u, y=6v) && \text{(到此步驟得 2分)} \\ &= \int\int_{u^2+v^2=1} \left[(2u)^2+(6v)^2\right] \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\, dv \quad (\text{let } u=r\cos\theta, v=r\sin\theta) \\ & && \text{(到此步驟得 4分)} \\ &= \int_{\theta=0}^{2\pi}\int_{r=0}^1 \left[(2r\cos\theta)^2+(6r\sin\theta)^2\right] 12\, r\, dr\, d\theta && \text{(到此步驟得 6分)} \\ &= 120\pi. && \text{(到此步驟得 7分)} \end{aligned}$$

7. (14 pts) (a) (7 pts) Find the volume of the region $\mathcal{R}$, where $\mathcal{R}$ is bounded by the surfaces $z = 0$, $x = 0$, $x = y^2$, and $z = 1 - y^2$.



(b) (7 pts) Let $\mathcal{R}$ be the part of the solid ball $x^2+y^2+z^2 \le 1$ which is above the cone $z = \sqrt{\dfrac{x^2+y^2}{3}}$. Evaluate $\iiint_{\mathcal{R}} (x^2+y^2)dV$.

Solution:

(a) Notice that $\mathcal{R}$, as seen from the given figure, can be expressed as the region under the graph of the function $f(x,y) = 1 - y^2$ over the plane region $\mathcal{D}$ (in the xy-plane) given by

$$\mathcal{D} = \left\{-1 \le y \le 1 \,,\; 0 \le x \le y^2\right\} .$$

(suitable description of the region $\mathcal{R}$: 3 points)

Therefore, the volume in question is

$$V = \iint_{\mathcal{D}} \left(1 - y^2\right) \, dA \qquad \text{(integral representing } V\text{: 1 point)}$$

$$= \int_{-1}^{1} \int_{0}^{y^2} \left(1 - y^2\right) \, dx \, dy \qquad \text{(Fubini's Theorem: 1 point)}$$

$$= \int_{-1}^{1} y^2\left(1 - y^2\right) \, dy \qquad \text{(computation: 1 point)}$$

$$= 2\int_{0}^{1} \left(y^2 - y^4\right) \, dy = 2\left(\frac{1}{3} - \frac{1}{5}\right) = \frac{4}{15} \, . \qquad \text{(answer: 1 point)}$$

(b) In spherical coordinated, $x^2+y^2 = (\rho\sin\phi\cos\theta)^2 + (\rho\sin\phi\sin\theta)^2 = \rho^2\sin^2\phi$. It implies that

$$\rho\cos\phi = \sqrt{\frac{x^2+y^2}{3}} = \frac{\rho}{\sqrt{3}}\sin\phi \Rightarrow \sqrt{3}\cos\phi = \sin\phi \Rightarrow \phi = \frac{\pi}{3}.$$

Thus $\mathcal{R} = \{(\rho,\phi,\theta) : 0 \le \rho \le 1, \, 0 \le \phi \le \frac{\pi}{3}, \, 0 \le \theta \le 2\pi\}$ (2 points).

$$\begin{aligned}
\iiint_{\mathcal{R}} (x^2+y^2)dV &= \iiint_{\mathcal{R}} \rho^4 \sin^3\phi d\rho d\phi d\theta \\
&= \int_0^{2\pi}\int_0^{\pi/3}\int_0^1 \rho^4\sin^3\phi d\rho d\phi d\theta \text{ (2 points)} \\
&= 2\pi \int_0^{\frac{\pi}{3}} [\frac{1}{5}\rho^5]_0^1 \sin^3\phi d\phi \\
&= \frac{2\pi}{5}\int_0^{\frac{\pi}{3}} \sin^3\phi d\phi \text{ (1 points)} \\
&= \frac{2\pi}{5}\int_0^{\frac{\pi}{3}} (1-\cos^2\theta)(-d\cos\theta) \\
&= \frac{2\pi}{5}[\frac{1}{3}\cos^3\theta - \cos\theta]_0^{\pi/3} \text{ (1 points)} \\
&= \frac{\pi}{12} \text{ (1 points)}
\end{aligned}$$

8. (12 pts) A lamina with constant density ρ occupies the region

$$\mathcal{R} := \left\{ (x,y) \,\middle|\, 1 \le x^{\frac{5}{3}} y \le 9 \,,\, 1 \le xy \le 4,\; x > 0 \right\} .$$

Find the coordinates $(\bar{x}, \bar{y})$ of its center of mass.

Solution:

The coordinates of the center of mass are given by

$$\bar{x} = \frac{\rho}{m} \iint_{\mathcal{R}} x \, dA \quad \text{and} \quad \bar{y} = \frac{\rho}{m} \iint_{\mathcal{R}} y \, dA \,, \text{ where } m := \rho \iint_{\mathcal{R}} dA \,.$$

The integrals are computed via a change of variables

$$s := x^{\frac{5}{3}} y \quad \text{and} \quad t := xy \quad \left(\iff \quad x = \left(\frac{s}{t}\right)^{\frac{3}{2}} \quad \text{and} \quad y = \frac{t^{\frac{5}{2}}}{s^{\frac{3}{2}}} \right)$$

with the corresponding Jacobian determinant given by

$$\frac{\partial(x,y)}{\partial(s,t)} = \left(\frac{\partial(s,t)}{\partial(x,y)} \right)^{-1} = \begin{vmatrix} \frac{5}{3} x^{\frac{2}{3}} y & x^{\frac{5}{3}} \\ y & x \end{vmatrix}^{-1} = \left(\frac{2}{3} x^{\frac{5}{3}} y \right)^{-1} = \frac{3}{2s} \,.$$

(proper strategy (e.g. change of variables) for the computation: 3 points)

As the region $\mathcal{R}$ is mapped to the rectangle $\{(s,t) \,|\, 1 \le s \le 9 \,,\, 1 \le t \le 4\}$, the coordinates of the center of mass can now be computed as

$$\begin{aligned}
m &= \rho \iint_{\mathcal{R}} dA = \rho \int_1^9 \int_1^4 \left| \frac{3}{2s} \right| dt \, ds = \frac{3\rho}{2} (4-1) \ln \left| \frac{9}{1} \right| = 9\rho \ln 3 \,, \\
\bar{x} &= \frac{\rho}{m} \iint_{\mathcal{R}} x \, dA = \frac{\rho}{m} \int_1^9 \int_1^4 \left(\frac{s}{t}\right)^{\frac{3}{2}} \left| \frac{3}{2s} \right| dt \, ds = \frac{3\rho}{2m} \int_1^4 \frac{1}{t^{\frac{3}{2}}} \, dt \cdot \int_1^9 s^{\frac{1}{2}} \, ds \\
&= \frac{3\rho}{2m} \left(\frac{2}{1^{\frac{1}{2}}} - \frac{2}{4^{\frac{1}{2}}} \right) \cdot \frac{2}{3} \left(9^{\frac{3}{2}} - 1^{\frac{3}{2}} \right) \\
&= \frac{26}{9 \ln 3} \,, \\
\bar{y} &= \frac{\rho}{m} \iint_{\mathcal{R}} y \, dA = \frac{\rho}{m} \int_1^9 \int_1^4 \frac{t^{\frac{5}{2}}}{s^{\frac{3}{2}}} \left| \frac{3}{2s} \right| dt \, ds = \frac{3\rho}{2m} \int_1^4 t^{\frac{5}{2}} \, dt \cdot \int_1^9 \frac{1}{s^{\frac{5}{2}}} \, ds \\
&= \frac{3\rho}{2m} \cdot \frac{2}{7} \left(4^{\frac{7}{2}} - 1^{\frac{7}{2}} \right) \cdot \frac{2}{3} \left(\frac{1}{1^{\frac{3}{2}}} - \frac{1}{9^{\frac{3}{2}}} \right) \\
&= \frac{2^2 \cdot 13 \cdot 127}{3^5 \cdot 7 \ln 3} \,.
\end{aligned}$$

(correct execution of the strategy (e.g. correct use of the change-of-variable formula): 3 points)

(computations for m, $m\bar{x}$ and $m\bar{y}$: 2+2+2 points)