1. (12%) Determine whether each of the following series is divergent, conditionally convergent, or absolutely convergent?

(a) (4%) $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n}{n(1+\ln n)}$. (b) (4%) $\displaystyle\sum_{n=1}^{\infty}(-1)^{n-1}\sin\left(\frac{1}{n^2}\right)$. (c) (4%) $\displaystyle\sum_{n=1}^{\infty}(-1)^n\frac{\pi^n n!}{n^n}$.

Solution:

(a) To see whether the series is convergent absolutely, we consider $\displaystyle\sum_{n=1}^{\infty}\left|\frac{(-1)^n}{n(1+\ln n)}\right| = \sum_{n=1}^{\infty}\frac{1}{n(1+\ln n)}$.

Now if we use the integral test for $\displaystyle\sum_{n=1}^{\infty}a_n$, $a_n = \dfrac{1}{n(1+\ln n)}$ with $a_n = f(n)$, $f(x) = \dfrac{1}{x(1+\ln x)}$, we examine $\displaystyle\int_1^{\infty}\frac{dx}{x(1+\ln x)} \overset{u=1+\ln x}{=\!=\!=} \int_1^{\infty}\frac{du}{u}$

Here $\displaystyle\int_1^{\infty}\frac{du}{u} = \lim_{\alpha\to\infty}\int_1^{\alpha}\frac{du}{u} = \lim_{\alpha\to\infty}\ln|u|\Big|_1^{\alpha} = \lim_{\alpha\to\infty}\ln\alpha - 0 = \infty$ (做到此步驟得 1 分)

Thus, we conclude that series is not absolutely convergent, according to the integral test for series convergence. Note that in order to use the integral test, we need to check the following conditions.

(i) $f(x)$ is continuous.

Here $f(x) = \dfrac{1}{x(1+\ln x)}$, $x \ge 1$, which is continuous on its domain $[1,\infty)$ since both $\dfrac{1}{x}$ and $\dfrac{1}{1+\ln x}$ are.

(ii) $f(x)$ is positive, since $\dfrac{1}{x} > 0$ and $\dfrac{1}{1+\ln x} > 0$ for $x \in [1,\infty)$.

(iii) $f(x)$ is decreasing, since $f'(x) = \left(\dfrac{1}{x(1+\ln x)}\right)' = (-1)(x(1+\ln x))^{-2}\cdot(1+(1+\ln x)) < 0$.

(檢查完這三個條件後得第 2 分)

Now we use the alternating series test to see whether series is convergent conditionally. That is, we perform

(i) $a_n \ge 0$, $\forall n \ge 1$

Here $a_n = \dfrac{1}{n(1+\ln n)}$, since $n \ge 1$ and $1+\ln n \ge 1$, it follows that $a_n \ge 0$

(ii) $a_{n+1} \le a_n$ $\forall n \ge 1$

Here $a_{n+1} = \dfrac{1}{(n+1)(1+\ln(n+1))} < \dfrac{1}{n(1+\ln n)} = a_n$ since $n+1 > n$ and $\ln(n+1) > \ln n$ $\forall n \ge 1$.

(iii) $\displaystyle\lim_{n\to\infty}a_n = 0$

Here $\displaystyle\lim_{n\to\infty}\frac{1}{n(1+\ln n)} = 0$ since $\displaystyle\lim_{n\to\infty}n(1+\ln n) = \infty$.

Thus it is convergent conditionally.

(檢查完這三個條件後得到 4 分，其中錯一個的話就少給 1 分)

(b) To see whether the series is convergent absolutely, we consider $\displaystyle\sum_{n=1}^{\infty}\left|(-1)^{n-1}\sin\left(\frac{1}{n^2}\right)\right| = \sum_{n=1}^{\infty}\sin\left(\frac{1}{n^2}\right)$.

Now let $a_n = \sin\left(\dfrac{1}{n^2}\right)$ and take $b_n = \dfrac{1}{n^2}$, $n \ge 1$.

Then we consider $\displaystyle\lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty}\frac{\sin\left(\frac{1}{n^2}\right)}{\frac{1}{n^2}} = 1$.

Thus by the limit comparison test (註：前面都對並且知道要使用 the limit comparison test 可得 3 分), we know that the series $\displaystyle\sum_{n=1}^{\infty}a_n$ converges, because $\displaystyle\sum_{n=1}^{\infty}b_n$ converges (which is a p-series with $p = 2 > 1$), and it is convergent absolutely.(註： 完成可得 4 分)

NOTE: Apply test for alternations series only (+3 if it is correct)

(c) By ratio test,

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\pi(n+1)}{(n+1)(1+\frac{1}{n})^n}\right| = \lim_{n\to\infty}\left|\frac{\pi}{(1+\frac{1}{n})^n}\right| = \frac{\pi}{e}.$$

Since $\dfrac{\pi}{e} > 1$, the original series diverges by ratio test.

Grading policy:

- Use ratio test. The student can get this point by just writing down the key word "ratio test". (1pt)
- Attempt to compute the limit $\lim_{n\to\infty} |\frac{a_{n+1}}{a_n}|$ (1pt)
- Compute the limit $\frac{\pi}{e}$ correctly. (1pt)
- Use ratio test with $\frac{\pi}{e}$ to conclude it diverges. (1pt)

Other than the solution above, some other situation may happen. The grader may decide how many partial credits an answer worth. The following are some guidelines:

- Use other test. Basically, only "Test for divergence", "Root test" may lead to partial credits. "Integral test", "(Limit) Comparison test", "Alternating series" worth 0 points basically. However, the student may still written some valuable argument down.
- Use Sterling formula. If the student can state the formula correctly, it is possible to get full credit from "Test for divergence", "Root test", "Limit comparison test".

2. (16%) For each of the following power series, find the interval of convergence and the function represented by it.

(a) (8%) $\sum_{n=1}^{\infty}(-1)^n \frac{1}{n8^n}(x-8)^n$.

(b) (8%) $\sum_{n=0}^{\infty} n^2 x^n$. (Hint: You can use the fact $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$.)

Solution:

(a) Let $a_n = \frac{(-1)^n}{n8^n}(x-8)^n$. Then

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{|x-8|^{n+1}}{(n+1)8^{n+1}}\frac{n8^n}{|x-8|^n} = \frac{1}{8}|x-8|.$$

By the ratio test (1%), this series converges absolutely when $\frac{1}{8}|x-8| < 1$ or for $0 < x < 16$. (1%)

When $x=0$, we have $\sum_{n=1}^{\infty}\frac{(-1)^n}{n8^n}(-8)^n = \sum_{n=1}^{\infty}\frac{1}{n}$ which is divergent by the p-series for $p=1$. (1%)

When $x=16$, we have $\sum_{n=1}^{\infty}\frac{(-1)^n}{n8^n}(8)^n = \sum_{n=1}^{\infty}\frac{(-1)^n}{n}$ which is convergent by the Alternating Series Test. (1%)

Hence, the interval of convergence is $(0,16]$. (1%)

Since $\ln(1+x) = \sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^n}{n}$, (1%)

$$\begin{aligned}\sum_{n=1}^{\infty}(-1)^n\frac{1}{n8^n}(x-8)^n &= -\sum_{n=1}^{\infty}(-1)^{n-1}\frac{\left(\frac{x-8}{8}\right)^n}{n} \quad (1\%)\\ &= -\ln(1+\frac{x-8}{8}) \quad (1\%)\end{aligned}$$

(b) The radius of convergence $R=1$, since

$$\lim_{n\to\infty}\frac{n^2}{(n+1)^2} = 1.$$

Both $x=-1$ and $x=1$ yield the series $\sum_{n=0}^{\infty} n^2$, which diverges since $\lim_{n\to\infty} n^2 = \infty$ (or does not exist). Hence the interval of convergence is $(-1,1)$.

Compute by term-by-term differentiation

$$\frac{1}{(1-x)^2} = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{1-x}\right) = \sum_{n=1}^{\infty} nx^{n-1} \tag{1}$$

$$\frac{x}{(1-x)^2} = \sum_{n=0}^{\infty} nx^n \tag{2}$$

$$\frac{1+x}{(1-x)^3} = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{x}{(1-x)^2}\right) = \sum_{n=1}^{\infty} n^2x^{n-1} \tag{3}$$

$$\frac{x(1+x)}{(1-x)^3} = \sum_{n=0}^{\infty} n^2x^n. \tag{4}$$

Thus $f(x) = \frac{x(1+x)}{(1-x)^3}$.

Grading Suggestion..

- *Get 1/8 by showing the radius of convergence* $R=1$.
- *Get 1/8 by showing that the series diverges at* $x=-1$.
- *Get 1/8 by showing that the series diverges at* $x=1$.
- *Get 1/8 by showing that the interval of convergence is* $(-1,1)$.
- *Get 1/8 by obtaining each of* (1–4).

Remark: Students are allowed to obtain (4) *by different methods, say by operating the identity* $(\frac{1}{1-x})'' = \sum_{n=2}^{\infty} n(n-1)x^{n-2}$. *In such cases, the grading TAs can decide the allocation of 3/8 from obtaining* (1–3).

3. (14%) Let $f(x) = \int_0^x \frac{1}{\sqrt{1+t^2}}\, dt$.

(a) (6%) Write down the Maclaurin series for $f(x)$ and find its radius of convergence.

(b) (8%) Approximate $f(\frac{1}{2})$ correct to within 0.01.

Solution:

(a)

$$\frac{1}{\sqrt{1+t^2}} = (1+t^2)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} \binom{-\frac{1}{2}}{n} t^{2n}, \text{ for } |t^2| < 1 \qquad \text{(2 pt)}$$

$$f(x) = \int_0^x \frac{1}{\sqrt{1+t^2}} dt = \int_0^x \sum_{n=0}^{\infty} \binom{-\frac{1}{2}}{n} t^{2n} dt = \sum_{n=0}^{\infty} \int_0^x \binom{-\frac{1}{2}}{n} t^{2n} dt = \sum_{n=0}^{\infty} \binom{-\frac{1}{2}}{n} \frac{1}{2n+1} x^{2n+1}, \text{ for } |t^2| < 1 \qquad \text{(2 pts)}$$

Radius of convergence is 1. (2 pts)

(1) $\because$ The binomial series $\sum_{n=0}^{\infty} \binom{-\frac{1}{2}}{n} x^{2n}$ has radius of convergence 1.
$\therefore$ The integral of the power series still has radius of convergence 1.

(2) By Ratio Test, $\left| \frac{\binom{-\frac{1}{2}}{n+1} \frac{1}{2(n+1)+1} x^{2(n+1)+1}}{\binom{-\frac{1}{2}}{n} \frac{1}{2n+1} x^{2n+1}} \right| = \frac{n+\frac{1}{2}}{n+1} \cdot \frac{2n+1}{2n+3} |x|^2 \to |x|^2$ as $n \to \infty$.
Hence the power series converges if $|x| < 1$ and the power series diverges if $|x| > 1$. Therefore the radius of convergence is 1.

(b) $f(\frac{1}{2}) = \sum_{n=0}^{\infty} \binom{-\frac{1}{2}}{n} \frac{1}{2n+1} (\frac{1}{2})^{2n+1} = \frac{1}{2} + \sum_{n=1}^{\infty} (-1)^n \frac{1 \cdot 3 \cdots (2n-1)}{2^n \cdot n!(2n+1)} \frac{1}{2^{2n+1}}$

$$\text{(2 pts)} \leftarrow \begin{cases} \text{Let } b_0 = \frac{1}{2},\ b_n = \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2^{3n+1} \cdot n!(2n+1)} \text{ for } n \geq 1.\ f(\frac{1}{2}) = \sum_{n=0}^{\infty} (-1)^n b_n. \\ \frac{b_{n+1}}{b_n} = \frac{2n+1}{2^3 \cdot (n+1)} \times \frac{2n+1}{2n+3} < \frac{1}{4} < 1 \\ \{b_n\} \text{ is decreasing and } \lim_{n \to \infty} b_n = 0 \end{cases}$$

(2 pts) $\leftarrow$ Hence $\left| f(\frac{1}{2}) - \sum_{n=0}^{k} (-1)^n b_n \right| < b_{k+1}$ by the alternating series estimation theorem.

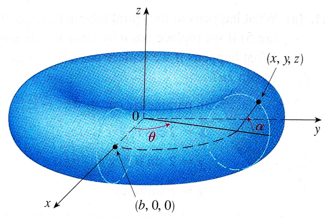
(2 pts) $\leftarrow$ With direct computation, we have $b_2 = \frac{1 \cdot 3}{2^7 \cdot 2!5} = \frac{3}{1280} < 0.01$

Hence we can approximate $f(\frac{1}{2})$ by $b_0 - b_1 = \frac{1}{2} - \frac{1}{6}\frac{1}{8} = \frac{23}{48}$ (2 pts)

4. (16%) Let S be a toroidal shell (not a solid torus!) obtained by rotating about the z-axis the circle C in the plane $y=0$ of radius a centred at $(b,0,0)$, where $0<a<b$. It can be parametrized by

$$\mathbf{r}(\theta,\alpha)=\langle(b+a\cos\alpha)\cos\theta\,,\,(b+a\cos\alpha)\sin\theta\,,\,a\sin\alpha\rangle\ ,\quad 0\le\theta,\alpha\le 2\pi\ .$$

Suppose that the temperature at each point of the surface is proportional to its distance from the plane $z=0$, i.e., the temperature $T(x,y,z)$ for every point $(x,y,z)\in S$ is given by $T(x,y,z)=\lambda|z|$ for some constant $\lambda>0$.



(a) (6%) Find the average temperature along the circle C which is $\dfrac{\int_C T\ ds}{\int_C 1\ ds}$.

(b) (10%) Find the area of S, $A(S)$. Then evaluate the average temperature of the toroidal shell S which is $\dfrac{\iint_S T\ dS}{A(S)}$.

Solution:

Step-by-step solution:

In part (a) we need two things, $\displaystyle\int_C T\ ds$ and $\displaystyle\int_C 1\ ds$.

First we parametrize C. The circle lies on the toroidal shell and it corresponds to the $\theta=0$.

$$\mathbf{r}_C(t)=\mathbf{r}(0,t)=\langle b+a\cos t,0,a\sin t\rangle\ ,\quad 0\le t\le 2\pi.$$

The orientation of C doesn't matter because we are evaluating scalar line integrals.

Evaluate $\displaystyle\int_C 1\ ds$. (it is the arc length, so we could also just use circumference.)

$$\int_C 1\ ds=\int_0^{2\pi}|\mathbf{r}_C'(t)|\ dt=\int_0^{2\pi}\sqrt{(-a\sin t)^2+0+(a\cos t)^2}\ dt=\int_0^{2\pi}|a|\ dt=2\pi a$$

Evaluate $\displaystyle\int_C T\ ds$.

$$\int_C T\ ds=\int_0^{2\pi}T(\mathbf{r}_C(t))\ |\mathbf{r}_C'(t)|\ dt=\int_0^{2\pi}(\lambda|a\sin t|)(a)\ dt=\lambda a^2\int_0^{2\pi}|\sin t|\ dt$$

We pause the evaluation to deal with the absolute value sign. Formally we should split the integral into pieces where $\sin t$ is positive and $\sin t$ is negative.

$$\lambda a^2\int_0^{2\pi}|\sin t|\ dt=\lambda a^2\left(\int_0^{\pi}\sin t\ dt+\int_{\pi}^{2\pi}(-\sin t)\ dt\right)=\lambda a^2\left(\Big[-\cos t\Big]_0^{\pi}+\Big[\cos t\Big]_{\pi}^{2\pi}\right)=4\lambda a^2$$

Alternatively by symmetry we could have used

$$\int_0^{2\pi}|\sin t|\ dt=2\int_0^{\pi}|\sin t|\ dt=4\int_0^{\pi/2}|\sin t|\ dt$$

Combine our results. The average temperature along the circle C is equal to $\dfrac{2\lambda a}{\pi}$. That concludes 4(a).

In part (b) we also need to evaluate two surface integrals, $\displaystyle\iint_S T\ dS$ and $\displaystyle\iint_S 1\ dS$.

First we parametrize S. The parametrization is given in the problem.

$$\mathbf{r}(\theta,\alpha)=\langle(b+a\cos\alpha)\cos\theta,(b+a\cos\alpha)\sin\theta,a\sin\alpha\rangle\ ,\quad 0\le\theta,\alpha\le 2\pi.$$

The orientation of the surface S doesn't matter because we are evaluating scalar surface integrals.

Before we evaluate $\iint_S 1\ dS$, this time we evaluate dS first.

$$dS = |\mathbf{r}_\theta \times \mathbf{r}_\alpha|\ d\theta\ d\alpha$$

The two partial derivatives are

$$\mathbf{r}_\theta(\theta, \alpha) = \langle -(b + a\cos\alpha)\sin\theta, (b + a\cos\alpha)\cos\theta, 0\rangle$$

and

$$\mathbf{r}_\alpha(\theta, \alpha) = \langle (-a\sin\alpha)\cos\theta, (-a\sin\alpha)\sin\theta, a\cos\alpha\rangle.$$

The cross product is

$$\mathbf{r}_\theta \times \mathbf{r}_\alpha = \langle a(b + a\cos\alpha)\cos\theta\cos\alpha, a(b + a\cos\alpha)\sin\theta\cos\alpha, a(b + a\cos\alpha)\sin\alpha\rangle$$

Simplify before we find $|\mathbf{r}_\theta \times \mathbf{r}_\alpha|$.

$$|\mathbf{r}_\theta \times \mathbf{r}_\alpha| = a(b + a\cos\alpha)|\langle\cos\theta\cos\alpha, \sin\theta\cos\alpha, \sin\alpha\rangle| = a(b + a\cos\alpha)$$

Note that $0 < a < b$ is used to guarantee $a(b + a\cos\alpha) > 0$.

Evaluate the surface area of S.

$$\iint_S 1\ dS = \int_0^{2\pi}\int_0^{2\pi} |\mathbf{r}_\theta \times \mathbf{r}_\alpha|\ d\theta\ d\alpha = \int_0^{2\pi}\int_0^{2\pi} a(b + a\cos\alpha)\ d\theta\ d\alpha = 4ab\pi^2$$

Evaluate $\iint_S T\ dS$.

$$\iint_S T(\mathbf{r}(\theta, \alpha)\ dS = \int_0^{2\pi}\int_0^{2\pi} (\lambda|a\sin\alpha|)(a(b + a\cos\alpha))\ d\theta\ d\alpha = 2\pi\lambda a^2 \int_0^{2\pi} (b + a\cos\alpha)|\sin\alpha|\ d\alpha$$

Similar to part (a), we divide the integral into pieces.

$$2\pi\lambda a^2 \int_0^{2\pi} (b + a\cos\alpha)|\sin\alpha|\ d\alpha = 2\pi\lambda a^2 \left(\int_0^{\pi} (b + a\cos\alpha)\sin\alpha\ d\alpha - \int_\pi^{2\pi} (b + a\cos\alpha)\sin\alpha\ d\alpha\right) = 8\pi\lambda a^2 b$$

Combine our results to get the average temperature on S: $\dfrac{2\lambda a}{\pi}$. This concludes 4(b).

Points distribution.

(2%) Parametrize C and the formula $ds = |\mathbf{r}'(t)|\ dt$.

(2%) The formula $dS = |\mathbf{r}_\theta \times \mathbf{r}_\alpha|\ d\theta\ d\alpha$.

(1%) Compute $|\mathbf{r}'(t)|$. No work needed.

(3%) Compute $|\mathbf{r}_\theta \times \mathbf{r}_\alpha|$.

(1%) The circumference $\int_C 1\ ds$. No work needed.

(2%) Evaluate $\int_C T\ ds$. 1 point for dealing with the absolute value.

(2%) Evaluate $\iint_S 1\ dS$.

(3%) Evaluate $\iint_S T\ dS$. 1 point for dealing with the absolute value.

Examples.

If they just show $\int_C T\ ds$, $\iint_S 1\ dS$, $\iint_S T\ dS$, then check to see if they show steps that imply understandings of the formula.

If student does not use the formula at all and integrated without $|\mathbf{r}'(t)|$ or $|\mathbf{r}_\theta \times \mathbf{r}_\alpha|$, then the most they can get is 8 points. (more if they decided to compute those but didn't use them in integrals)

Forgetting the absolute value in the function T will cause the student to lose 2 points, 1 in (a) and 1 in (b). They lose more points if they just write "zero by symmetry" since we wouldn't be able to give them points for knowing how to integrate correctly.

A mistake in $|\mathbf{r}_\theta \times \mathbf{r}_\alpha|$ may result in very complicated integrals. A mistake in evaluating that and not finishing the integrals will cause them to lose at least 3 points (1 calculation mistake and 2 unfinished integrals) and they lose more if they didn't deal with $|z|$ correctly or integrate as far as they can.

No point value is assigned to the final answer. If they did everything except stating the final average temperature, then they lose 1 point (even if they forgot in both (a) and (b)).

If an integral is set up the wrong way, the grader can decide whether it counts as a mistake in formula or a mistake in evaluating the integral. For example, a vector line integral would be correct parametrization of C but wrong formula, no magnitude, and wrong setup, at least 3 points off.

Similarly if a vector surface integral is used, they lose a lot of points.

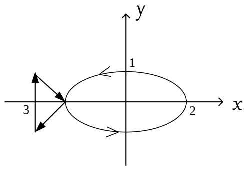
No points will be given for (a) or for (b) by using symmetry to state that they would have the same average. The problem is stated in the sense that the students need to verify this fact. (Unless they prove a formal theorem that covers general surfaces of revolutions and general functions.)

5. (14%) Consider the vector field $\mathbf{F}(x,y) = P(x,y)\,\mathbf{i} + Q(x,y)\,\mathbf{j} = \dfrac{x+3y}{x^2+y^2}\,\mathbf{i} + \dfrac{-3x+y}{x^2+y^2}\,\mathbf{j}$.

(a) (4%) Show that $\mathbf{F}$ is conservative on the half plane $D = \{(x,y) | x < 0\}$.

(b) (5%) Compute $\int_{C_0} \mathbf{F}\cdot d\mathbf{r}$, where C_0 is the unit circle $x^2+y^2=1$ oriented counterclockwise. Is $\mathbf{F}$ conservative on $\mathbb{R}^2\backslash\{(0,0)\}$?

(c) (5%) Compute $\int_C \mathbf{F}\cdot d\mathbf{r}$, where C is a piecewise smooth path consisting of the ellipse $\dfrac{x^2}{4}+y^2=1$ and the triangle formed by the lines $x=-3$, $x+y=-2$, and $y-x=2$. The orientation of C is shown in the figure.



Solution:

(a) (Method 1) Compute $\dfrac{\partial P}{\partial y}$ and $\dfrac{\partial Q}{\partial x}$

$\dfrac{\partial P}{\partial y} = \dfrac{3x^2-2xy-3y^2}{(x^2+y^2)^2}$, $\dfrac{\partial Q}{\partial x} = \dfrac{3x^2-2xy-3y^2}{(x^2+y^2)^2}$. (1 pt for each partial derivative)

$D = \{(x,y)|x<0\}$ is open and simply-connected. (1 pt)

Because P, Q have continuous first-order partial derivatives on D and $P_y(x,y) = Q_x(x,y)$ on D, $\mathbf{F} = P\mathbf{i} + Q\mathbf{j}$ is conservative on D. (1 pt)

(Method 2)

$f(x,y) = \dfrac{1}{2}\ln(x^2+y^2) - 3\tan^{-1}(\dfrac{y}{x})$ has continuous first-order partial derivatives on D.

And $\dfrac{\partial}{\partial x} f(x,y) = \dfrac{x+3y}{x^2+y^2} = P(x,y)$, $\dfrac{\partial}{\partial y} f(x,y) = \dfrac{-3x+y}{x^2+y^2} = Q(x,y)$.

Hence $\nabla f = \mathbf{F}(x,y)$ on D which shows that $\mathbf{F}$ is conservative on D.

(b) Parametrize C_0 as $\mathbf{r}(\theta) = (\cos\theta, \sin\theta)$, $0 \le \theta \le 2\pi$ (1 pt)

$$\int_{C_0} \mathbf{F}\cdot d\mathbf{r} = \int_0^{2\pi} \mathbf{F}(\mathbf{r}(\theta))\cdot\mathbf{r}'(\theta)d\theta \qquad \text{(1 pt)}$$

$$\left.\begin{aligned} &= \int_0^{2\pi} (\cos\theta + 3\sin\theta, -3\cos\theta+\sin\theta)\cdot(-\sin\theta,\cos\theta)d\theta \\ &= \int_0^{2\pi} -3d\theta = -6\pi \end{aligned}\right\} \qquad \text{(2 pts)}$$

C_0 is a closed curve in $\mathbb{R}^2\backslash\{(0,0)\}$ but $\int_{C_0} \mathbf{F}\cdot d\mathbf{r} = -6\pi \neq 0$.

Hence we conclude that $\mathbf{F}$ is not conservative on $\mathbb{R}^2\backslash\{(0,0)\}$. (1 pt)

NOTE:
1. 直接使用 Green's Theorem 在單位圓內部得 0 pt.
2. 只有結論 "$\mathbf{F}$ is not conservative on $\mathbb{R}^2\backslash\{(0,0)\}$", 沒有正確的理由，不給結論的 1 pt.

(c) Let $C = C_1 \cup C_2$, where C_1 is the ellipse and C_2 is the triangle.

Because C_2 is a closed path in D and $\mathbf{F}$ is conservative on D, we know that $\int_{C_2} \mathbf{F}\cdot d\mathbf{r} = 0$. (2 pt)

Apply Green's Theorem on the region Ω which is inside C_1 and outside the unit circle C_0.
$\partial\Omega = C_1 \cup (-C_0)$.
$P(x,y)$ and $Q(x,y)$ have continuous partial derivatives on Ω.
Hence $\int_{\partial\Omega} \mathbf{F}\cdot d\mathbf{r} = \iint_\Omega Q_x - P_y dA = \iint_\Omega 0dA = 0$
Therefore $\int_{\partial\Omega} \mathbf{F}\cdot d\mathbf{r} = \int_{C_1} \mathbf{F}\cdot d\mathbf{r} - \int_{C_0} \mathbf{F}\cdot d\mathbf{r} = 0$
$\Rightarrow \int_{C_1} \mathbf{F}\cdot d\mathbf{r} = \int_{C_0} \mathbf{F}\cdot d\mathbf{r} = -6\pi$
(3 pts)

$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_{C_1} \mathbf{F}\cdot d\mathbf{r} + \int_{C_2} \mathbf{F}\cdot d\mathbf{r} = -6\pi + 0 = -6\pi$

NOTE: 如果使用橢圓和三角形的參數式直接以線積分定義計算，也算對。但是沒有算完積分，只列出積分式只各得 1 pt.

6. (12%) Let $f(x, y, z) = x + xy + yz + zx$, and $g(x, y, z) = x + 2y + 3z$.

(a) (5%) Show by direct calculation that $\operatorname{curl}(f\,\nabla g) = \nabla f \times \nabla g$.

(b) (7%) Find $\iint_S (\nabla f \times \nabla g) \cdot \mathbf{n}\,dS$ where S is the surface $z = \sqrt{4 - x^2 - y^2}$, and $\mathbf{n}$ is the unit normal on S pointing upwards.

Solution:

(a) $\nabla f = (1 + y + z,\ x + z,\ x + y)$, $\nabla g = (1, 2, 3) \Rightarrow$
$\nabla f \times \nabla g = (x - 2y + 3z,\ x - 2y - 3z - 3,\ z + 2y - x + 2)$ (2%)

$\operatorname{curl}(f\nabla g) = \nabla \times [f(x, y, z)(1, 2, 3)] = (3f_y - 2f_z,\ f_z - 3f_x,\ 2f_x - f_y)$
$= (x - 2y + 3z,\ x - 2y - 3z - 3,\ z + 2y - x + 2) = \nabla f \times \nabla g$ (3%)

(b) $\iint_S (\nabla f \times \nabla g) \cdot \mathbf{n}\,dS = \iint_S \operatorname{curl}(f\nabla g) \cdot \mathbf{n}\,dS = \oint_C (f\nabla g) \cdot d\mathbf{r}$
where C is the circle $x^2 + y^2 = 4$ traced counterclockwise (2%)

$\Rightarrow \oint_C (f\nabla g) \cdot d\mathbf{r} = \oint_C x(1 + y)(dx + 2dy)$ (2%)
$= 4\int_0^{2\pi} (1 + 2\sin\theta)\cos\theta(-\sin\theta + 2\cos\theta)d\theta$ (2%)
$= 8\int_0^{2\pi} \cos^2\theta\,d\theta = 8\pi$ (1%)

(Method 2)
For those employing direct surface integral:
$\iint_S (\nabla f \times \nabla g) \cdot \mathbf{n}\,dS =$
$\iint_{x^2+y^2\le 4} (\nabla f \times \nabla g) \cdot \dfrac{(x, y, z)}{2}\sqrt{1 + z_x^2 + z_y^2}\,dxdy$ (1%)
$\iint_{x^2+y^2<4} \dfrac{x^2 - 2y^2 + z^2 - xy + 2xz - yz - 3y + 2z}{z}\,dxdy$ (1%)
$= \int_0^{2^-}\int_0^{2\pi} \left[\dfrac{r^2(\cos^2\theta - 2\sin^2\theta)}{\sqrt{4 - r^2}} + 2 + \sqrt{4 - r^2}\right] rdrd\theta$ (2%)
$= \pi\int_0^{2^-} \left[(4 + 2\sqrt{4 - r^2}) - \dfrac{r^2}{\sqrt{4 - r^2}}\right] rdr$ (1%)
$= 8\pi\int_0^{\frac{\pi}{2}} \left[(1 + \cos\theta)\sin 2\theta - \sin^3\theta\right] d\theta = 8\pi$ (2%)

7. (16%) Consider the unit disk

$$S_1 = \left\{(x,y,z) \in \mathbb{R}^3 \,\middle|\, x^2+y^2 \le 1 \,,\ z = 0\right\}$$

and the half cone

$$S_2 = \left\{(x,y,z) \in \mathbb{R}^3 \,\middle|\, 2z = 1 - \sqrt{x^2+y^2} \,,\ z \ge 0\right\} \,,$$

and let $S = S_1 \cup S_2$ be the closed surface of a cone with the positive (outward) orientation. Both S_1 and S_2 are endowed with the induced orientation from S.

(a) (6%) Let $\mathbf{F} = \langle 0 \,,\, y^2 \,,\, z - 2yz \rangle$. Find $\iint_S \mathbf{F} \cdot d\mathbf{S}$.

(b) (10%) Let $\mathbf{G} = \mathbf{F} + \mathbf{E}$, where $\mathbf{E} = \left\langle \dfrac{x}{(x^2+y^2+z^2)^{\frac{3}{2}}} \,,\, \dfrac{y}{(x^2+y^2+z^2)^{\frac{3}{2}}} \,,\, \dfrac{z}{(x^2+y^2+z^2)^{\frac{3}{2}}} \right\rangle$ defined on $\mathbb{R}^3 \backslash \{(0\,,0\,,0)\}$.

Find $\iint_{S_2} \mathbf{G} \cdot d\mathbf{S}$. (Note that the integral is only over S_2.)

Solution:

(a) Let E be the solid cone bounded by S. As $\mathbf{F}$ is of class $\mathscr{C}^1$ on a neighbourhood of E, we can apply the Divergence theorem to obtain

$$\begin{aligned}\iint_S \mathbf{F} \cdot d\mathbf{S} &= \iiint_E \operatorname{div} \mathbf{F} \, dV = \iiint_E (2y + 1 - 2y)\, dV \\ &= V(E) = \int_0^{2\pi} \int_0^1 \int_0^{\frac{1}{2} - \frac{r}{2}} r \, dz \, dr \, d\theta = \pi \left[\frac{r^2}{2} - \frac{r^3}{3} \right]_0^1 = \frac{\pi}{6} \,.\end{aligned}$$

(correct application of the divergence theorem: 3 points)

(computation of $\operatorname{div} \mathbf{F}$: 1 point)

($V(E)$: 2 points)

Alternative method The integral can be computed directly. First, we have

$$\iint_{S_1} \mathbf{F} \cdot d\mathbf{S} = \iint_{S_1} \mathbf{F} \cdot (-\mathbf{k}) \, dA = 0 \quad \left(\text{since } (\mathbf{F} \cdot \mathbf{k})|_{S_1} = 0\right) \,. \qquad \text{(1 point)}$$

To compute $\iint_{S_2} \mathbf{F} \cdot d\mathbf{S}$, take the parametrisation of S_2 given by

$$\mathbf{r}(r,\theta) := \left\langle r\cos\theta \,,\, r\sin\theta \,,\, \frac{1-r}{2} \right\rangle \,,\ 0 \le r \le 1 \,,\ 0 \le \theta \le 2\pi \,, \qquad \text{(1 point)}$$

which yields

$$\mathbf{r}_r \times \mathbf{r}_\theta = \left\langle \frac{r\cos\theta}{2} \,,\, \frac{r\sin\theta}{2} \,,\, r \right\rangle \,. \qquad \text{(1 point)}$$

Therefore,

$$\begin{aligned}\iint_{S_2} \mathbf{F} \cdot d\mathbf{S} &= \int_0^{2\pi} \int_0^1 \mathbf{F} \cdot \mathbf{r}_r \times \mathbf{r}_\theta \, dr \, d\theta \qquad \text{(1 point)} \\ &= \int_0^{2\pi} \int_0^1 \left(\left(r^2 \sin^2\theta\right)\left(\frac{r\sin\theta}{2}\right) + \frac{1-r}{2}(1 + 2r\sin\theta)\, r \right) dr \, d\theta \\ &= \int_0^{2\pi} \int_0^1 \left(\frac{r^3}{2} \sin^3\theta + \frac{(1-r)\, r}{2} + r^2(1-r)\sin\theta \right) dr \, d\theta \\ &= -\frac{1}{8} \int_{\theta=0}^{\theta=2\pi} \left(1 - \cos^2\theta\right) d\cos\theta + \frac{\pi}{6} + 0 = \frac{\pi}{6} \,. \qquad \text{(1 point)}\end{aligned}$$

As a result,

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_{S_1} \mathbf{F} \cdot d\mathbf{S} + \iint_{S_2} \mathbf{F} \cdot d\mathbf{S} = 0 + \frac{\pi}{6} = \frac{\pi}{6} \,. \qquad \text{(1 point)}$$

(b) Since $\iint_S \mathbf{F} \cdot d\mathbf{S} = \frac{\pi}{6}$ by the result of (a) and

$$\iint_{S_2} \mathbf{F} \cdot d\mathbf{S} = \iint_S \mathbf{F} \cdot d\mathbf{S} - \iint_{S_1} \mathbf{F} \cdot d\mathbf{S} = \frac{\pi}{6} - 0 = \frac{\pi}{6} \,,$$

($\iint_{S_2} = \iint_S - \iint_{S_1}$: 2 points) (value of $\iint_{S_1} \mathbf{F} \cdot d\mathbf{S}$: 1 point)

it remains to compute $\iint_{S_2} \mathbf{E} \cdot d\mathbf{S}$.

Note that $\operatorname{div} \mathbf{E} = 0$ on $\mathbb{R}^3 \setminus \{(0,0,0)\}$. Let

$$H := \left\{ (x,y,z) \in \mathbb{R}^3 \,\middle|\, x^2 + y^2 + z^2 = 1 \,,\ z \geq 0 \right\}$$

be the upper unit hemisphere centred at the origin endowed with the upward orientation. Notice that H and S_2 bounded a solid region R with its positively (outwardly) oriented boundary surface ∂R given by $H - S_2$ up to orientation, and E is $\mathscr{C}^1$ on a neighbourhood of R. The Divergence theorem can thus be applied to obtain

$$\iint_H \mathbf{E} \cdot d\mathbf{S} - \iint_{S_2} \mathbf{E} \cdot d\mathbf{S} = \iint_{H - S_2} \mathbf{E} \cdot d\mathbf{S} = \iiint_R \operatorname{div} \mathbf{E} \, dV = 0 \,,$$

which infers that

$$\begin{aligned}
\iint_{S_2} \mathbf{E} \cdot d\mathbf{S} &= \iint_H \mathbf{E} \cdot d\mathbf{S} \\
&= \iint_H \mathbf{E} \cdot \frac{\nabla\left(x^2 + y^2 + z^2 - 1\right)}{\left|\nabla\left(x^2 + y^2 + z^2 - 1\right)\right|} \, dS \\
&= \iint_H \langle x \,,\ y \,,\ z \rangle \cdot \langle x \,,\ y \,,\ z \rangle \, dS && \left(\text{since } \left(x^2 + y^2 + z^2\right)\big|_H = 1\right) \\
&= \iint_H 1 \, dS = 2\pi \,.
\end{aligned}$$

(computation of $\iint_{S_2} \mathbf{E} \cdot d\mathbf{S}$ (choice of H + use of the Divergence thm.): 3+3 points)

As a result,

$$\iint_{S_2} \mathbf{G} \cdot d\mathbf{S} = \iint_{S_2} \mathbf{F} \cdot d\mathbf{S} + \iint_{S_2} \mathbf{E} \cdot d\mathbf{S} = \frac{\pi}{6} + 2\pi = \frac{13\pi}{6} \,. \qquad \text{(linearity: 1 point)}$$

Alternative computation of $\iint_{S_2} \mathbf{E} \cdot d\mathbf{S}$. The integral can be computed directly with the parametrisation of S_2 given in the solution to Question (a).

($\mathbf{r}$ and $\mathbf{r}_r \times \mathbf{r}_\theta$: 1+1 points)

Indeed,

$$\begin{aligned}
\iint_{S_2} \mathbf{E} \cdot d\mathbf{S} &= \int_0^{2\pi} \int_0^1 \mathbf{E} \cdot \mathbf{r}_r \times \mathbf{r}_\theta \, dr \, d\theta && \text{(1 point)} \\
&= \int_0^{2\pi} \int_0^1 \frac{\frac{r^2 \cos^2\theta}{2} + \frac{r^2 \sin^2\theta}{2} + \frac{r(1-r)}{2}}{\left(r^2 + \left(\frac{1-r}{2}\right)^2\right)^{\frac{3}{2}}} \, dr \, d\theta \\
&= \frac{5\sqrt{5}}{2} \int_0^{2\pi} \int_0^1 \frac{r}{\left(\left(\frac{5}{2}\left(r - \frac{1}{5}\right)\right)^2 + 1\right)^{\frac{3}{2}}} \, dr \, d\theta \\
&\overset{\frac{5}{2}\left(r - \frac{1}{5}\right) = \tan\phi}{=} 5\sqrt{5}\pi \int_{\tan\phi = -\frac{1}{2}}^{\tan\phi = 2} \frac{\left(\frac{2}{5}\tan\phi + \frac{1}{5}\right) \frac{2}{5}\sec^2\phi}{\left(\sec^2\phi\right)^{\frac{3}{2}}} \, d\phi \\
&= \frac{2}{\sqrt{5}}\pi \int_{\phi_0}^{\phi_0 + \frac{\pi}{2}} (2\sin\phi + \cos\phi) \, d\phi && \left(\phi_0 := \tan^{-1}\left(-\frac{1}{2}\right)\right) \\
&= \frac{2}{\sqrt{5}}\pi \left[-2\cos\phi + \sin\phi \right]_{\phi_0}^{\phi_0 + \frac{\pi}{2}} \\
&= 2\pi \left[-\cos(\phi - \phi_0) \right]_{\phi_0}^{\phi_0 + \frac{\pi}{2}} = 2\pi \,, && \text{(3 points)}
\end{aligned}$$

in which the facts $\sin\phi_0 = -\dfrac{1}{\sqrt{5}}$ and $\cos\phi_0 = \dfrac{2}{\sqrt{5}}$ are used.