1071微乙01-05班期中考解答和評分標準

1. (14%) 求下列極限。

(a) (7%) $\lim_{x\to 0}\dfrac{x\sin x}{1-\cos x}$.

(b) (7%) $\lim_{x\to\infty}\left(1+\dfrac{2}{x}\right)^{3x}$. (Hint: $\lim_{x\to\infty}(1+\dfrac{1}{x})^x = e$)

Solution:

(a)

sol 1:

$$\lim_{x\to 0}\frac{x\sin\ x}{1-\cos\ x} = \lim_{x\to 0}\frac{x\sin\ x(1+\cos\ x)}{(1-\cos\ x)(1+\cos\ x)} = \lim_{x\to 0}\frac{x\sin\ x(1+\cos\ x)}{\sin^2\ x} = \lim_{x\to 0}\frac{x(1+\cos\ x)}{\sin\ x}(3pt)$$

$$since \lim_{x\to 0}\frac{x}{\sin\ x} = 1(2pt); \lim_{x\to 0}(1+\cos) = 2$$

$$\lim_{x\to 0}\frac{x(1+\cos\ x)}{\sin\ x} = \lim_{x\to 0}\frac{x}{\sin\ x}\times\lim_{x\to 0}(1+\cos) = 1\times\ 2 = 2(2pt)$$

sol 2:

$$\lim_{x\to 0}\frac{x\sin\ x}{1-\cos\ x} = \lim_{x\to 0}\frac{2x\sin\ \frac{x}{2}\cos\ \frac{x}{2}}{1-(1-2\sin^2\ \frac{x}{2})} = \lim_{x\to 0}\frac{2x\sin\ \frac{x}{2}\cos\ \frac{x}{2}}{2\sin^2\ \frac{x}{2}} = \lim_{x\to 0}\frac{x\cos\ \frac{x}{2}}{\sin\ \frac{x}{2}} = \lim_{x\to 0}\frac{(2\times\ \frac{x}{2})\cos\ \frac{x}{2}}{\sin\ \frac{x}{2}}(3pt)$$

$$since \lim_{x\to 0}\frac{\frac{x}{2}}{\sin\ \frac{x}{2}} = 1(2pt); \lim_{x\to 0}(\cos\ \frac{x}{2}) = 1$$

$$\lim_{x\to 0}\frac{(2\times\ \frac{x}{2})\cos\ \frac{x}{2}}{\sin\ \frac{x}{2}} = 2\times\ \lim_{x\to 0}\frac{\frac{x}{2}}{\sin\ \frac{x}{2}}\times\ \lim_{x\to 0}\cos\ \frac{x}{2} = 2\times\ 1\times\ 1 = 2(2pt)$$

sol 3: (by L'Hôpital's rule)

$$\lim_{x\to 0}\frac{x\sin\ x}{1-\cos\ x}(\frac{0}{0}) = \lim_{x\to 0}\frac{x\cos\ x+\sin\ x}{\sin\ x}(2pt)(\frac{0}{0}) = \lim_{x\to 0}\frac{\cos\ x+\cos\ x-x\sin\ x}{\cos\ x}(1pt) = 2(1pt)$$

note: you would get 4 points if you use L'Hôpital's rule totally correct and get the right answer.

(b)

$$\lim_{x\to\infty}(1+\frac{2}{x})^{(3x)} = \lim_{x\to\infty}[(1+\frac{2}{x})^{(\frac{x}{2})}]^6(5pt)$$

let $y=\frac{x}{2}$, when $x\to\infty, y\to\infty$

$$\lim_{y\to\infty}[(1+\frac{1}{y})^y]^6 = e^6(2pt)$$

note1: you will lose 2 points if you write $\lim_{x\to\infty}(1+\frac{2}{x})^x = e^2$ without satisfied explanation.
note2: you will get 2 points if you use exponential law correctly.

2. (14%) 求下列函數的一階導數。

(a) (7%) $f(x) = \tan(2^x)$.

(b) (7%) $f(x) = (\sin x)^x$.

Solution:

(a)

$$(\tan\ x)' = \sec^2\ x, (2^x)' = 2^x \times \ln(2)$$

let $y = 2^x$

$$ln(y) = x \times \ln(2)$$

$$\frac{y'}{y} = ln(2)$$

$$y' = y \times \ln(2) = 2^x \times \ln(2)$$

$$\frac{df}{dx} = \sec^2\ (2^x) \times\ (2^x)' = \sec^2\ (2^x)(3pt) \times\ (2^x) \times \ln(2)(4pt)$$

note: 1 point will be given if you did wrong on finding the derivative of 2^x, but you tried to tell me that chain rule should be used.

(b) Step 1:

$$\mathrm{f(x) = \sin x^x = e^{\ln \sin x^x} = e^{x \ln \sin x}}$$

Step 2:

$$\mathrm{f'(x) = (x \ln(\sin x))' e^{x \ln(\sin x)} = (\ln \sin x + x \cdot \frac{\cos x}{\sin x}) \cdot \sin x^x}$$

criterion

1 point : Something wrong in result "and" all calc process.
2 points : Something wrong in result "but" calc in first step was right .
3 points : Something wrong in result "but" overall calc processes were right.
4 or 5 points : Something wrong in result "but" all calc process were right.
7 points : win the score.

Note: You get only 1 point if giving $f'(x) = x\ \sin x^{x-1}$

or a number of derivations to this result,it seems kind of tedious...

3. (12%) 求方程式 $\tan^{-1}\dfrac{y}{x} = 2xy - 2y^2 + \dfrac{\pi}{4}$ 在 $(1,1)$ 的一階與二階導數，$\dfrac{dy}{dx}$, $\dfrac{d^2y}{dx^2}$.

Solution:

Step1: $tan^{-1}y/x = 2xy - 2y^2 + \pi/4$
Step2: do implicit differentiation

$$(\frac{y}{x})' \cdot \frac{1}{(\frac{y}{x})^2 + 1} = (2y + 2xy') - 4yy'$$

$$\frac{y - xy}{(\frac{y}{x})^2 + 1}\frac{x^2}{y^2 + x^2} = 2y + (2x - 4y)y'$$

$$y' = \frac{\frac{y-xy}{(\frac{y}{x})^2+1}\frac{x^2}{y^2+x^2} - 2y}{2x - 4y}$$

$$y'_{(1,1)} = 1$$

Step3: Do implicit diffierentiation again(from 2nd line in Step2)

$$\frac{(x^4 \cdot (y - xy))' \cdot (x^2 + y^2)^2 - (x^4 \cdot (y - xy)) \cdot ((x^2 + y^2)^2)'}{(x^2 + y^2)^4} = 2y' + (2 - 4y')y' + (2x - 4y)y''$$

$$y'' = \frac{\frac{(x^4\cdot(y-xy))'\cdot(x^2+y^2)^2 - (x^4\cdot(y-xy))\cdot((x^2+y^2)^2)'}{(x^2+y^2)^4} - (4 - 4y')y'}{2x - 4y}$$

$$y''_{(1,1)} = 0$$

criterion

1 point: Giving any function not related to these questions.
2 points: Differentiate equations partially right but answer was wrong.
3 points: Differentiate equations partially right and detailedly but answers was wrong.
4 points: Differentiate overall equations right but answer was wrong.
6 points: Differentiate overall equations right and answer was right.

4. (8%) 利用線性逼近估計 $\sqrt[4]{10004}$.

Solution:

Let $f(x) = \sqrt[4]{x}$, the tangent line at $x = a$ is $L(x) = f'(a)(x-a) + f(a) = \dfrac{1}{4a^{\frac{3}{4}}}(x-a) + \sqrt[4]{a}$

Linear approximation gives that $f(x) \approx L(x)$ when $x \approx a$.

Consider $x = 10004$, $a = 10000$ in the case.

Then $f(10004) \approx L(10004) = \dfrac{1}{4 \cdot 10000^{\frac{3}{4}}}(10004 - 10000) + \sqrt[4]{10000} = 10.001$

Scoring :

(1) Differentiation of $\sqrt[4]{x}$ (2 point)

(2) Computing the tangent line (4 point)

(3) Final answer (2 point)

(4) One will get full credit even if one approximate the value by another line if the error is smaller than 1.

5. (8%) 當 $0 < x < 1$ 時，證明 $-\ln(1-x) > x + \frac{1}{2}x^2$.

Solution:

Define the function $f(x) = x + \frac{1}{2}x^2 + \ln(1-x)$ on the interval $[0,1)$

Notice $f'(x) = 1 + x - \frac{1}{1-x} = \frac{-x^2}{1-x} < 0$, $\forall\ x \in (0,1)$, hence $f(x)$ is decreasing on $(0,1)$

But $f(0) = 0$, hence $f(x) < 0$, $\forall\ x \in (0,1)$. That is, $x + \frac{1}{2}x^2 < -\ln(1-x)$, $\forall\ x \in (0,1)$.

Scoring :

(1) Differentiation of $f(x)$ and the verification of the monotone property (either increasing or decreasing, depend on the function) (4 point)

(2) Using the fact $f(0) = 0$ or Mean value theorem to prove $f(x) < 0$ (2 point)

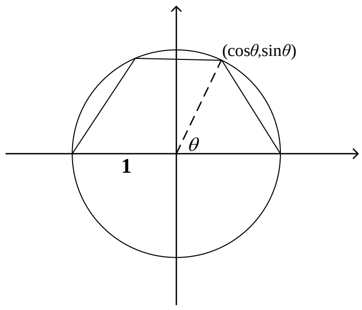
(3) Final result. (2 point)

6. (14%) 如圖，在單位圓內，考慮一個以直徑為底的內接梯形。

(a) (4%) 把內接梯形的面積寫成 θ 的函數。我們稱這個面積函數是 $f(\theta)$.

(b) (7%) 求 $f(\theta)$ 在 $0 < \theta < \frac{\pi}{2}$ 間的極值候選點。

(c) (3%) 求 $f(\theta)$ 在 $0 \le \theta \le \frac{\pi}{2}$ 間的最大值。



Solution:

(a) $0 < \theta < \frac{\pi}{2}$

$$\begin{aligned} \text{上底} &= 2\cos\theta & \ldots 1\% \\ \text{下底} &= 2 & \ldots 1\% \\ \text{高} &= \sin\theta & \ldots 1\% \\ \text{面積} &= f(\theta) = \sin\theta + \sin\theta\cos\theta & \ldots 1\% \end{aligned}$$

(b)

$$f'(\theta) = 2\cos^2\theta + \cos\theta - 1 \qquad \ldots 2\%$$

$$f'(\theta) = 0 \Rightarrow \cos\theta = -1, \frac{1}{2}(-1\text{不合}) \qquad \ldots 2\%$$

$$\therefore \cos\theta = \frac{1}{2} \quad , \quad \theta = \frac{\pi}{3} \qquad \ldots 3\%$$

(c)

$$0 \le \theta < \frac{\pi}{3}, \quad f'(\theta) > 0$$

$$\frac{\pi}{3} < \theta \le \frac{\pi}{2}, \quad f'(\theta) < 0 \qquad \ldots 1\%$$

$$\therefore f(\frac{\pi}{3}) = \frac{3\sqrt{3}}{4}\text{為最大值} \qquad \ldots 2\%$$

7. (14%) 找出函數 $y = f(x) = \sqrt{4x^2 + x}$ 的斜漸近線，其中 $x \le -\frac{1}{4}$ 或 $x \ge 0$。

Solution:

(14%) 找出函數 $y = f(x) = \sqrt{4x^2 + x}$ 的斜漸近線，其中 $x \le -\frac{1}{4}$ 或 $x \ge 0$。

(3pts) 找正無限大的斜率

$$\lim_{x\to\infty} \frac{f(x)}{x} = \lim_{x\to\infty} \frac{\sqrt{4x^2+x}}{x} = \lim_{x\to\infty} \frac{|x|\sqrt{4x+\frac{1}{x}}}{x} = \lim_{x\to\infty} \sqrt{4+\frac{1}{x}} = 2$$

(3pts) 找負無限大的斜率

$$\lim_{x\to-\infty} \frac{f(x)}{x} = \lim_{x\to-\infty} \frac{\sqrt{4x^2+x}}{x} = \lim_{x\to-\infty} \frac{|x|\sqrt{4x+\frac{1}{x}}}{x} = \lim_{x\to-\infty} -\sqrt{4+\frac{1}{x}} = -2$$

(3pts) 找正無限大的k

$$\lim_{x\to\infty} f(x) - 2x = \lim_{x\to\infty} \sqrt{4x^2+x} - 2x = \lim_{x\to\infty} \frac{4x^2+x-(4x^2)}{\sqrt{4x^2+x}+2x}$$

$$= \lim_{x\to\infty} \frac{x}{\sqrt{4x^2+x}+2x} = \lim_{x\to\infty} \frac{x}{|x|\sqrt{4+\frac{1}{x}}+2x} = \lim_{x\to\infty} \frac{1}{\sqrt{4+\frac{1}{x}}+2} = \frac{1}{4}$$

(3pts) 找負無限大的k

$$\lim_{x\to-\infty} f(x) + 2x = \lim_{x\to-\infty} \sqrt{4x^2+x} + 2x = \lim_{x\to-\infty} \frac{4x^2+x-(4x^2)}{\sqrt{4x^2+x}-2x}$$

$$= \lim_{x\to-\infty} \frac{x}{\sqrt{4x^2+x}-2x} = \lim_{x\to-\infty} \frac{x}{|x|\sqrt{4+\frac{1}{x}}-2x} = \lim_{x\to-\infty} \frac{1}{-\sqrt{4+\frac{1}{x}}-2} = -\frac{1}{4}$$

(1pts) 找正無限大的斜漸近線 $y = 2x + \frac{1}{4}$

(1pts) 找負無限大的斜漸近線 $y = -2x - \frac{1}{4}$

評分標準:

1.如果有寫出 $m = \lim_{x\to\infty} \frac{f(x)}{x}$ 正負得分開寫且分開討論，如果寫在一起則得一分，如果分開寫則各得一分。

2.如果有寫出 $k = \lim_{x\to\infty} f(x) - mx$ 正負得分開寫且分開討論，如果寫在一起則得一分，如果分開寫則各得一分。

8. (16%) 若 $f(x) = 3\ln(x^2-1) - 4x$.

(a) $f(x)$ 的定義域是 ____________________.

(b) $f'(x) =$ ____________________.

$f(x)$ 在 ____________________ (區間)遞增。$f(x)$ 在 ____________________ (區間)遞減。

(c) $f''(x) =$ ____________________.

$f(x)$ 在 ____________________ (區間)凹向上(若存在的話)

$f(x)$ 在 ____________________ (區間)凹向下(若存在的話)

(d) $f(x)$ 在 $x =$ ____________________ 有極大值 ____________________(若存在的話)

$f(x)$ 在 $x =$ ____________________ 有極小值 ____________________(若存在的話)

(e) $y = f(x)$ 的垂直漸近線為 ____________________

(f) 畫出 $y = f(x)$ 之圖形

Solution:

(a). $x^2 - 1 > 0 \Rightarrow x \in (-\infty, -1) \cup (1, \infty)$ (1%)

(b). $f'(x) = 3 \cdot \dfrac{2x}{x^2-1} - 4 = \dfrac{-4x^2+6x+4}{x^2-1} = \dfrac{-2(2x+1)(x-2)}{(x-1)(x+1)}$ (2%)

The function is increasing on $(1, 2)$ (1%) and decreasing on $(-\infty, -1) \cup (2, \infty)$ (1%).

(c). $f''(x) = \dfrac{(x^2-1)(-8x+6) - (-4x^2+6x+4)(2x)}{(x^2-1)^2} = \dfrac{-6(x^2+1)}{(x^2-1)^2} < 0$ (2%)

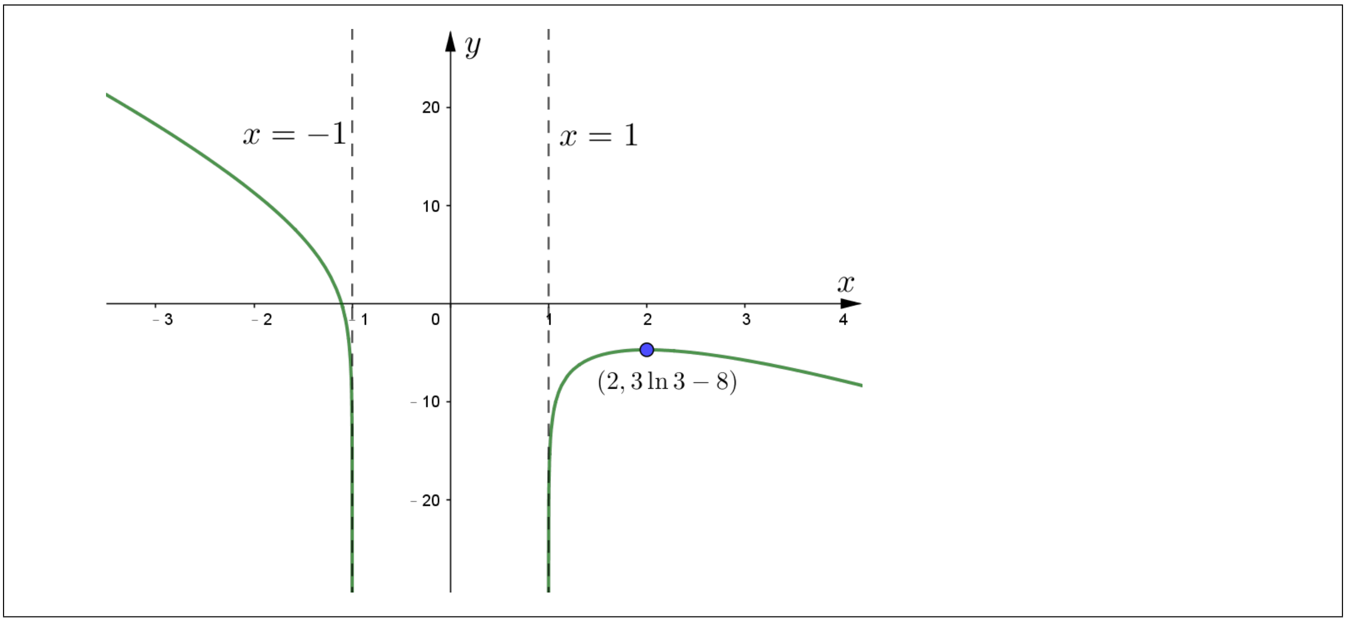
Concave up: none. (1%)

Concave down: $(-\infty, -1) \cup (1, \infty)$. (1%)

(d). By (b), the function has maximum at $x = 2$ ($f(2) = 3\ln 3 - 8$, 1%) and no minimum (1%).

(e). $x = 1$ and $x = -1$ (2%).

(f). 1% for increasing and decreasing intervals, 1% for the concavity, and 1% for the extremes.


y
x
$x = -1$
$x = 1$
20
10
−10
−20
−3
−2
−1
0
1
2
3
4
$(2, 3\ln 3 - 8)$