1071微乙01-05班期末考解答和評分標準

1. (12%) 若一連續函數 f 滿足以下的方程式：對於所有的 $x>0$, $\int_a^{x^2} \frac{f(\sqrt{t})}{t} dt + 2 = x$. 求 $f(x)$, $x>0$ 和常數 $a>0$.

Solution:

Sol 1:

令 $F(x) = \int_a^x \frac{f(\sqrt{t})}{t}\, dt$ ，則根據微積分基本定理有 $F'(x) = \frac{f(\sqrt{x})}{x}$ 。

原式即為 $F(x^2)+2 = x$ ，對等式兩邊微分得到 $2xF'(x^2) = 1 \Longrightarrow 2x\cdot\frac{f(\sqrt{x^2})}{x^2} = 1 \Longrightarrow \underline{f(x) = \frac{x}{2}}$ 。

$\int \frac{f(\sqrt{t})}{t}\, dt = \int \frac{\frac{\sqrt{t}}{2}}{t}\, dt = \int \frac{1}{2}\cdot t^{-\frac{1}{2}}\, dt = \frac{1}{2}\cdot 2t^{\frac{1}{2}} + C = \sqrt{t}+C$ ， 代回原式得 $\left(\sqrt{t}\right)\Big|_a^{x^2} + 2 = x \Longrightarrow x - \sqrt{a} + 2 = x \Longrightarrow \sqrt{a} = 2,\ \underline{a=4}$ 。

Sol 2:

令 $G(x) = \int_{a(x)}^{b(x)} g(t)\, dt$ ，則 $G'(x) = g(b(x))\cdot b'(x) - g(a(x))\cdot a'(x)$ 。

取 $a(x) = a,\ b(x) = x^2,\ g(t) = \frac{f(\sqrt{t})}{t}$ ，則 $G(x) = \int_a^{x^2}\frac{f(\sqrt{t})}{t}\, dt = x-2$ 。

對等式兩邊微分有 $G'(x) = \frac{f(\sqrt{x^2})}{x^2}\cdot b'(x) - \frac{f(\sqrt{a})}{a}\cdot a'(x) = \frac{f(\sqrt{x^2})}{x^2}\cdot 2x - \frac{f(\sqrt{a})}{a}\cdot 0 = (x-2)' = 1$ ，移項整理得 $\underline{f(x) = \frac{x}{2}}$ 。

另外，若在原式中直接代 $x=\sqrt{a}$ 會得到 $\int_a^{\sqrt{a^2}}\frac{f(\sqrt{t})}{t}\, dt + 2 = 0+2 = \sqrt{a}$ ，故 $\sqrt{a} = 2,\ \underline{a=4}$ 。

評分標準:

(1) 求出 $f(x)$ 的部分佔 9 分，若只有答案正確給 3 分。求出 a 的部分佔 3 分，若只有答案正確給 1 分。

(2) 方法二中的公式使用錯誤，導致求出錯誤的 $f(x)$ ，扣2到4分。

(3) 沒求 a 的值，或者因為 $f(x)$ 求錯而得到不正確的 a 值，一律扣 3 分。

(4) 只有寫出方法二的公式者給 4 分；只有寫出微積分基本定理的敘述者給 2 分。

(5) 計算錯誤、錯字酌情扣0到3分。

(6) 邏輯謬誤(例如令 $F(x) = \int_0^x \frac{f(\sqrt{t})}{t}\, dt$)和符號重複(例如令 $f(x) = \int_a^x \frac{f(\sqrt{t})}{t}\, dt$)等，不影響計算的細節者不扣分。

(7) 其他情形亦酌情給分。

2. (24%) 計算下列積分。

(a) (8%) $\int_0^1 \sin^{-1} x \, dx.$ (b) (8%) $\int \tan^4 x \, dx.$ (c) (8%) $\int \frac{1}{x^4-1} dx.$

Solution:

(a) 由分部積分得

$$\begin{aligned}\int \sin^{-1} x \, dx &= x\sin^{-1} x - \int \frac{x}{\sqrt{1-x^2}}\, dx \qquad (3分)\\ &= x\sin^{-1} x + \frac{1}{2}\int \frac{1}{\sqrt{u}}\, du \quad (變數變換\ u = 1-x^2) \qquad (2分)\\ &= x\sin^{-1} x + \sqrt{1-x^2} + C \qquad (2分)\end{aligned}$$

因此

$$\int_0^1 \sin^{-1}\, dx = \sin^{-1}(1) - 1 = \frac{\pi}{2} - 1 \qquad (1分)$$

(b)

$$\begin{aligned}\int \tan^4 x \, dx &= \int \tan^2 x\,(\sec^2 x - 1)\, dx\\ &= \int \tan^2 x \sec^2 x \, dx - \int \tan^2 x \, dx \qquad (2分)\\ &= \int \tan^2 x \, d(\tan x) - \int \sec^2 x - 1 \, dx\\ &= \frac{\tan^3 x}{3} - \tan x + x + C \qquad (3分/2分/1分)\end{aligned}$$

(c) 由部分分式知

$$\frac{1}{x^4-1} = \frac{Ax+B}{x^2+1} + \frac{C}{x-1} + \frac{D}{x+1} \qquad (1分)$$

經過計算可得

$$A = 0, \quad B = -\frac{1}{2}, \quad C = \frac{1}{4}, \quad D = -\frac{1}{4} \qquad (4分)$$

因此

$$\begin{aligned}\int \frac{1}{x^4-1} &= \int \frac{-\frac{1}{2}}{x^2+1} + \int \frac{\frac{1}{4}}{x-1} + \int \frac{-\frac{1}{4}}{x+1}\\ &= -\frac{1}{2}\tan^{-1} x + \frac{1}{4}\ln|x-1| - \frac{1}{4}\ln|x+1| + C' \qquad (3分)\end{aligned}$$

註1.: 不定積分需加常數 C
註2.: $\ln|x-1|$ 及 $\ln|x+1|$ 之絕對值不可省略

3. (12%) 求瑕積分 $\int_0^\infty e^{-x}\sin x\ dx$.

Solution:

Method I .

Consider $\int e^{-x}\sin x\,dx$

$= -e^{-x}\sin x + \int e^{-x}\cos x\theta \qquad \ldots 5\%$

$= -e^{-x}\sin x - e^{-x}\cos x - \int e^{-x}\sin x\,dx \qquad \ldots 3\%$

$\Longrightarrow \int e^{-x}\sin x\,dx = \dfrac{-e^{-x}(\sin x+\cos x)}{2} + C \qquad \ldots 1\%$

$\int_0^\infty e^{-x}\sin x\,dx = \lim_{b\to\infty}\int_0^b e^{-x}\sin x\,dx = \lim_{b\to\infty}\dfrac{-e^{-x}(\sin x+\cos x)}{2}\Big|_0^b = \dfrac{1}{2} \qquad \ldots 3\%$

Method 2 .

$\int e^{-x}\sin x\,dx = -e^{-x}\sin x + \int e^{-x}\cos x\,dx \qquad \ldots 5\%$

$\int e^{-x}\sin x\,dx = -e^{-x}\cos x - \int e^{-x}\cos x\,dx \qquad \ldots 0\%$

Add up two equation and divided by two then we get,

$\int e^{-x}\sin x\,dx = \dfrac{-e^{-x}(\sin x+\cos x)}{2} + C \qquad \ldots 4\%$

$\int_0^\infty e^{-x}\sin x\,dx = \lim_{b\to\infty}\int_0^b e^{-x}\sin x\,dx = \lim_{b\to\infty}\dfrac{-e^{-x}(\sin x+\cos x)}{2}\Big|_0^b = \dfrac{1}{2} \qquad \ldots 3\%$

4. (8%) 求曲線 $y = \dfrac{e^{2x} + e^{-2x}}{4}$，$x$ 由 0 到 1 的曲線長。

Solution:

弧長 $= \int_0^1 \sqrt{1 + (\frac{dy}{dx})^2} dx = \int_0^1 \sqrt{1 + (\frac{e^{2x} - e^{-2x}}{2})^2} dx \ldots 4pts$

$= \int_0^1 \sqrt{\frac{e^{4x} + 2 + e^{-4x}}{4}} dx = \int_0^1 \sqrt{(\frac{e^{2x} + e^{-2x}}{2})^2} dx = \int_0^1 \frac{e^{2x} + e^{-2x}}{2} dx \ldots 2pts$

$= \frac{1}{4}(e^{2x} - e^{-2x})|_0^1 = \frac{e^{2x} - e^{-2x}}{4} \ldots 2pts$

5. (12%) 求以圓盤 $x^2 + (y-2)^2 \leq 1$ 繞 x 軸旋轉所形成之環體體積。

Solution:

Method 1:

We know y to represent these two functions.

$$y_1 = \sqrt{1-x^2} + 2 \quad \text{upper bound}$$

and

$$y_2 = -\sqrt{1-x^2} + 2 \quad \text{lower bound} \quad (2pt)$$

Hence we use the volume of y_1 to cut the y_2 of volume around the x-axis

$$\int_{-1}^{1} \pi y_1^2 dx - \int_{-1}^{1} \pi y_2^2 dx = \int_{-1}^{1} \pi[(1-x^2+4\sqrt{1-x^2}+4) - (1-x^2-4\sqrt{1-x^2}+4)]dx \quad (2pt)$$

$$= \int_{-1}^{1} 8\pi\sqrt{1-x^2}dx \quad (\text{Let x} = \sin(\theta)) \quad (2pt)$$

$$= 8\int_{-\frac{\pi}{2}}^{-\frac{\pi}{2}} \cos^2(\theta)d\theta = 8\pi\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1+\cos(2\theta)}{2}d\theta \quad (3pt)$$

$$= 8\pi[\frac{\theta}{2} + \frac{\sin(2\theta)}{4}]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = 16\pi(\frac{\pi}{4} + 0) = 4\pi^2 \quad (3pt)$$

Method 2:

Center of gravity is (0,2) (2pt)

Use Pappus theorem (8pt)

$$2\pi\overline{y} * \pi * (r)^2 = 4\pi^2 \quad (2pt)$$

6. (16%)

(a) (8%) 寫下 $(1-x^2)^{-\frac{1}{2}}$ 在 $x=0$ 的泰勒展式。

(b) (8%) 由 (a) 推導出 $\sin^{-1} x$ 在 $x=0$ 的泰勒展式，並寫出非 0 的前三項。

Solution:

(a) Binomial expansion: $(1+t)^a = \sum_{n=0}^{\infty} \mathrm{C}_n^a t^n$.

Let $t=-x^2$ and $a=\frac{-1}{2}$

$$(1-x^2)^{\frac{-1}{2}} = \sum_{n=0}^{\infty} \mathrm{C}_n^{\frac{-1}{2}} (-x^2)^n$$
$$= \sum_{n=0}^{\infty} (-1)^n \mathrm{C}_n^{\frac{-1}{2}} x^{2n}$$

(b) Since $(\sin^{-1} x)' = \frac{1}{\sqrt{1-x^2}}$, we can use the result of (a) to find the Taylor expansion of $\sin^{-1} x$

$$\sin^{-1} x = \int_0^x \frac{1}{\sqrt{1-t^2}} dt$$
$$= \int_0^x \Big(\sum_{n=0}^{\infty} (-1)^n \mathrm{C}_n^{\frac{-1}{2}} t^{2n} \Big) dt$$
$$= \sum_{n=0}^{\infty} (-1)^n \mathrm{C}_n^{\frac{-1}{2}} \Big(\int_0^x t^{2n} dt \Big)$$
$$= \sum_{n=0}^{\infty} (-1)^n \mathrm{C}_n^{\frac{-1}{2}} \frac{1}{2n+1} x^{2n+1}$$

The first three terms that are non-zero are

$$(-1)^0 \mathrm{C}_0^{\frac{-1}{2}} \frac{1}{1} x^1 + (-1)^1 \mathrm{C}_1^{\frac{-1}{2}} \frac{1}{3} x^3 + (-1)^2 \mathrm{C}_2^{\frac{-1}{2}} \frac{1}{5} x^5$$
$$= x + (-1) \frac{-1}{2} \frac{1}{3} x^3 + (-1)^2 \frac{-1}{2} \frac{-3}{2} \frac{1}{5} x^5$$
$$= x + \frac{1}{6} x^3 + \frac{3}{40} x^5$$

Grading Criteria

(a) Write down binomial expansion get 4 points.
Write down the Taylor expansion correctly get the rest of 4 points.

Note that if you try to use the definition of Taylor series to solve this problem, you can get at most 4 points unless you write down the general form of the Taylor expansion correctly.

(b) Knowing the relation between $\sin^{-1} x$ and $\frac{1}{\sqrt{1-x^2}}$ get 2 points.
Write down the Taylor expansion of $sin^{-1}x$ get 3 points.
For the first three non-zero terms of $sin^{-1}x$, each term is 1 points.

Similar to (a), you can get at most 4 points if you try to use the definition of Taylor series to solve this problem.

7. (16%) 求下列的極限值。

(a) (8%) $\lim_{x\to 0} \frac{\sin(x^2)}{e^x - 1 - x}$.

(b) (8%) $\lim_{x\to\infty} (1+\sqrt{x})^{\frac{1}{\ln x}}$.

Solution:

(a) (method 1)

because $\sin(0^2) = 0,\ e^0 - 1 - 0 = 0$

by L'Hopital Rule

$$\lim_{x\to 0} \frac{\sin(x^2)}{e^x - 1 - x} = \lim_{x\to 0} \frac{(\sin(x^2))'}{(e^x - 1 - x)'} = \lim_{x\to 0} \frac{2x\cos(x^2)}{e^x - 1}$$

beacause $2 \cdot 0 \cdot \sin(0^2) = 0,\ e^0 - 1 = 0$

by L'Hopital Rule again

$$\lim_{x\to 0} \frac{2x\cos(x^2)}{e^x - 1} = \lim_{x\to 0} \frac{(2x\cos(x^2))'}{(e^x - 1)'} = \lim_{x\to 0} \frac{-4x^2\sin(x^2) + 2\cos(x^2)}{e^x}$$

$$= \frac{-4 \times 0^2 \times sin(0^2) + 2 \times cos(0^2)}{e^0} = 2$$

(method 2)

$$sinx = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots\ldots$$

$$sin(x^2) = x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} + \ldots\ldots$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots\ldots$$

$$\text{so } \lim_{x\to 0} \frac{\sin(x^2)}{e^x - 1 - x} = \lim_{x\to 0} \frac{x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} + \ldots\ldots}{(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots\ldots) - 1 - x} = \lim_{x\to 0} \frac{x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} + \ldots\ldots}{\frac{x^2}{2!} + \frac{x^3}{3!} + \ldots\ldots}$$

$$= \lim_{x\to 0} \frac{1 - \frac{x^4}{3!} + \frac{x^8}{5!} + \ldots\ldots}{\frac{1}{2!} + \frac{x^1}{3!} + \ldots\ldots} = 2$$

在 method 1 裡

寫出第一個羅必達得兩分，微分正確得兩分

寫出第二個羅必達得兩分，微分正確且答案正確得兩分

看得出來有使用卻沒有寫出”羅必達”者扣一分，計算錯誤扣一分

在method 2 裡

寫出 $\sin(x^2)$ 的泰勒展開式得三分

寫出 e^x 的泰勒展開式者得三分

算式正確且答案無誤得兩分

(b) $\lim_{x\to\infty} (1+\sqrt{x})^{\frac{1}{lnx}} = \lim_{x\to\infty} e^{ln((1+\sqrt{x})^{\frac{1}{lnx}})} = \lim_{x\to\infty} e^{\frac{ln(1+\sqrt{x})}{lnx}}$

because e^x is a continuous function

so $\lim_{x\to\infty} e^{\frac{ln(1+\sqrt{x})}{lnx}} = e^{\lim_{x\to\infty} \frac{ln(1+\sqrt{x})}{lnx}}$

and because $ln(1+\sqrt{\infty}) = \infty, ln(\infty) = \infty$

by L'Hopital Rule

$$\lim_{x\to\infty} \frac{\ln(1+\sqrt{x})}{\ln x} = \lim_{x\to\infty} \frac{\frac{1}{1+\sqrt{x}} \cdot \frac{1}{2\sqrt{x}}}{\frac{1}{x}} = \lim_{x\to\infty} \frac{x}{2\sqrt{x} + 2x} = \lim_{x\to\infty} \frac{1}{\frac{2}{\sqrt{x}} + 2} = \frac{1}{0+2} = \frac{1}{2}$$

$$\text{so } \lim_{x\to\infty} (1+\sqrt{x})^{\frac{1}{lnx}} = \lim_{x\to\infty} e^{ln((1+\sqrt{x})^{\frac{1}{lnx}})} = \lim_{x\to\infty} e^{\frac{ln(1+\sqrt{x})}{lnx}} = e^{\lim_{x\to\infty} \frac{ln(1+\sqrt{x})}{lnx}} = e^{\frac{1}{2}}$$

有成功將原式取 ln 或變成 e 的指數型態者得兩分

使用羅必達者得兩分

微分正確得兩分，最終答案正確得兩分

(看得出來有使用卻沒有寫出”羅必達”者，加上上題至多只扣一次一分)