There are 8 questions in this question set.
The total score of this question set is 100 points.
All questions should be answered. The use of calculator is not allowed.
Justify each step of your arguments as far as possible. **Time limit: 2.5 hours**

1. Let $\{a_n\}_{n=0}^{\infty}$ be a sequence defined by the recurrence relation

$$a_0 = 1 \quad \text{and} \quad a_{n+1} := a_n - \frac{a_n^2 - 2}{2a_n} = \frac{a_n}{2} + \frac{1}{a_n} \quad \text{for every } n \in \mathbb{N} \cup \{0\} .$$

(a) Assuming that $\{a_n\}_{n=0}^{\infty}$ is convergent, find its limit. (4 points)

(b) Show that

$$a_{n+1}^2 - 2 = \frac{(a_n^2 - 2)^2}{4a_n^2} < \frac{(a_n^2 - 2)^2}{4} \quad \text{for all } n \in \mathbb{N} .$$ (3 points)

(c) Argue that $\{a_n\}_{n=0}^{\infty}$ is convergent. (5 points)

(You may, after the exam, google "Babylonian method for computing square root" for more information about this sequence.)

Solution:

(a) It can be seen easily from induction that $a_n > 0$ for all $n \in \mathbb{N}$. Suppose that $\lim_{n\to\infty} a_n =: L \geq 0$. (argument for $L \geq 0$: 1 point)

Then

$$L^2 = \lim_{n\to\infty} a_{n+1}a_n = \lim_{n\to\infty} \left(\frac{a_n^2}{2} + 1\right) = \frac{L^2}{2} + 1 \quad \Rightarrow \quad L = \sqrt{2} \quad \text{or} \quad -\sqrt{2} \text{ (rejected)} .$$

Note that the limit of $a_{n+1}a_n$ instead of just a_{n+1} is considered in order to avoid having to deal with the possibility that $L = 0$ separately. (3 points)

(b) A direct computation shows that, for any $n \in \mathbb{N} \cup \{0\}$,

$$a_{n+1}^2 - 2 = \left(\frac{a_n}{2} + \frac{1}{a_n}\right)^2 - 2 = \left(\frac{a_n}{2} - \frac{1}{a_n}\right)^2 = \frac{(a_n - 2)^2}{4a_n^2} .$$ (1 point)

Since the right-hand-side is non-negative for any $n \in \mathbb{N} \cup \{0\}$, it follows that $a_n^2 \geq 2 > 1$ for all $n \in \mathbb{N}$. (1 point)

The strict inequality in the claim then follows accordingly. (1 point)

(c) It follows that

$$\left|a_{n+1}^2 - 2\right| < \frac{\left|a_n^2 - 2\right|^2}{4} < \frac{\left|a_{n-1}^2 - 2\right|^4}{4^3} < \cdots < \frac{\left|a_1^2 - 2\right|^{2^n}}{4^{2^n - 1}} = \frac{\left|\left(\frac{3}{2}\right)^2 - 2\right|^{2^n}}{4^{2^n - 1}} = \frac{1}{4^{2^{n+1} - 1}} .$$

(correct inequalities: 2 points)

The far-most expression on the right-hand-side converges to 0 as $n \to \infty$. By the Squeeze Theorem, $\left|a_n^2 - 2\right|$ converges to 0, which in turn implies that $a_n^2 - 2$ converges to 0 or a_n^2 converges to 2, as $n \to \infty$. As $x \mapsto \sqrt{x}$ is a continuous function for $x > 0$, one has $\lim_{n\to\infty} a_n = \sqrt{2}$. In particular, the limit exists. (3 points)

2. Determine if each of the following series is absolutely convergent, conditionally convergent or divergent. (5 points each, total: 15 points)

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{\cos n}{(3n-2)^{n+\frac{1}{2}}}$ (b) $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^n}{\ln(n!)}$ (c) $\displaystyle\sum_{n=1}^{\infty}(-1)^n\left(3^{n^{-2}}-1\right)$

Solution:

(a) Set $a_n := \frac{\cos n}{(3n-2)^{n+\frac{1}{2}}}$ and note that

$$|a_n| = \left|\frac{\cos n}{(3n-2)^{n+\frac{1}{2}}}\right| \le \frac{1}{|3n-2|^{n+\frac{1}{2}}} =: |b_n|$$

$$\text{and} \qquad \lim_{n\to\infty} |b_n|^{\frac{1}{n}} = \lim_{n\to\infty} \frac{1}{|3n-2|^{1+\frac{1}{2n}}} = 0 < 1 \,.$$

It follows that the series $\sum_{n=1}^{\infty} b_n$ is absolutely convergent by the Root Test and, in turn, the series $\sum_{n=1}^{\infty} a_n$ is *absolutely convergent* by the Comparison Test.

(correct use of tests: 4 points)

(conclusion: 1 point)

(b) Set $a_n := \frac{(-1)^n}{\ln(n!)}$ and note that

$$|a_n| = \frac{1}{\underbrace{\ln 1 + \ln 2 + \cdots + \ln n}_{n \text{ terms}}} > \frac{1}{n \ln n} \qquad \text{for } n \ge 2\,.$$

Since

$$\int_2^{\infty} \frac{dx}{x \ln x} = [\ln\ln x]_2^{\infty} = \infty\,,$$

the series $\sum_{n=2}^{\infty} \frac{1}{n\ln n}$ is divergent by the Integral Test and thus the series $\sum_{n=2}^{\infty} a_n$ is *not absolutely convergent* by the Comparison Test. However, the series is an alternating series. It can be seen that

$$|a_n| = \frac{1}{\ln(n!)} > \frac{1}{\ln((n+1)!)} = |a_{n+1}| \qquad \text{and} \qquad \lim_{n\to\infty} \frac{1}{\ln(n!)} = 0.$$

By the Alternating Series Test, $\sum_{n=2}^{\infty} a_n$ is convergent. As a result, the series is *conditionally convergent.*

(correct use of tests: 3 points)

(conclusion (convergent but not absolutely convergent): 2 points)

(c) Set $a_n := (-1)^n\left(3^{n^{-2}}-1\right) = (-1)^n\left(e^{\frac{\ln 3}{n^2}}-1\right)$ and note that

$$\lim_{n\to\infty} \frac{|a_n|}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{\sum_{k=1}^{\infty} \frac{1}{k!}\left(\frac{\ln 3}{n^2}\right)^k}{\frac{1}{n^2}} = \ln 3 < \infty\,.$$

As $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges as a p-series with $p = 2 > 1$, it follows from the Limit Comparison Test that the series $\sum_{n=1}^{\infty} a_n$ *converges absolutely.*

(correct use of tests: 4 points)

(conclusion: 1 point)

3. Consider the power series $\sum_{n=1}^{\infty}(-1)^{n-1}n^p(2x-1)^{2n}$.

   (a) Find all the possible radii and intervals of convergence as p varies. (9 points)

   (b) Identify the series with an elementary function f when $p=-1$. (4 points)

   (c) Approximate $\int_{0.5}^{0.55} f(x)dx$ up to an error less than 10^{-7}. (4 points)

Solution:

(a) Set $a_n := (-1)^{n-1}n^p(2x-1)^{2n}$. Note that

$$\frac{|a_{n+1}|}{|a_n|} = \frac{(n+1)^p\,|2x-1|^{2(n+1)}}{n^p\,|2x-1|^{2n}} = \left(1+\frac{1}{n}\right)^p |2x-1|^2 \to |2x-1|^2 \quad \text{as } n\to\infty\,.$$

By the Ratio Test, the series converges absolutely when $|2x-1|^2<1 \iff \left|x-\frac{1}{2}\right|<\frac{1}{2}$ and diverges when $\left|x-\frac{1}{2}\right|>\frac{1}{2}$, so the radius of convergence is $\frac{1}{2}$ for any real number p.

(correct use of tests: 3 points)

(radius: 1 point)

To determine the interval of convergence, it remains to check convergence at the points such that $2x-1=\pm 1$. In both cases, the series becomes $\sum_{n=1}^{\infty}(-1)^{n-1}n^p$ which is an alternating $(-p)$-series. Notice that

$$n^p \begin{cases} \to\infty & \text{when } n\to\infty \text{ and } p>0 \;\Rightarrow \text{series diverges by the Test for Divergence;} \\ \to 1\neq 0 & \text{when } n\to\infty \text{ and } p=0 \;\Rightarrow \text{series diverges by the Test for Divergence;} \\ \searrow 0 & \text{when } n\nearrow\infty \text{ and } p<0 \;\Rightarrow \text{series converges by the Alternating Series Test.} \end{cases}$$

It follows that the given power series converges at both $x=0$ and $x=1$ if and only if $p<0$. Therefore, the interval of convergence is $[0,1]$ when $p<0$ and is $(0,1)$ when $p\geq 0$.

(correct use of tests: 3 points)

(correctly distinguishing different cases of p: 1 point)

(intervals: 1 point)

(b) Since we have the power series expansion

$$\ln(1+x) = \sum_{n=1}^{\infty}\frac{(-1)^{n-1}x^n}{n} \quad \text{for } |x|<1\,,$$

it follows that the given power series when $p=-1$ is $\ln\left(1+(2x-1)^2\right) = \ln\left(4x^2-4x+2\right)$ on its interval of convergence $[0,1]$. (4 points)

(c) Notice that

$$\int_{0.5}^{0.55}\ln\left(4x^2-4x+2\right)dx = \int_{0.5}^{0.55}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}(2x-1)^{2n}}{n}dx = \sum_{n=1}^{\infty}\frac{(-1)^{n-1}(0.1)^{2n+1}}{2n(2n+1)}\,.$$

By the Alternating Series Estimate, the given integral can be approximated by

$$b_1-b_2 = \frac{10^{-3}}{6}-\frac{10^{-5}}{20} = 0.0001661\dot{6}$$

(correct approximation according to the choice of the bound for the error: 2 points)

with an error bounded by

$$b_3 = \frac{10^{-7}}{42} < 10^{-7}\,,$$

which is within the required range. (correct use of estimate: 2 points)

Cont.

4. Suppose a curve C (a conical helix) is given by the parametric equations

$$r = a\theta \quad \text{and} \quad z = b\theta \ ,$$

where r and θ are the polar coordinates in the (x, y)-plane, and a and b are positive constants such that $a^2 + b^2 = 1$.

(a) Write down a vector function $\mathbf{r}$ which represents the curve C. (3 points)

(b) Find the curvature function κ of C. (6 points)

Solution:

(a) $\mathbf{r}(\theta) := \langle a\theta\cos\theta\ ,\ a\theta\sin\theta\ ,\ b\theta\rangle$. (3 points)

(b) Note that

$$\begin{aligned}
\mathbf{r}'(\theta) &= \langle a\cos\theta - a\theta\sin\theta\ ,\ a\sin\theta + a\theta\cos\theta\ ,\ b\rangle\ ,\\
\left|\mathbf{r}'(\theta)\right| &= \sqrt{(a\cos\theta - a\theta\sin\theta)^2 + (a\sin\theta + a\theta\cos\theta)^2 + b^2}\\
&= \sqrt{a^2\theta^2 + a^2 + b^2} = \sqrt{a^2\theta^2 + 1}\ ,\\
\mathbf{r}''(\theta) &= \langle -2a\sin\theta - a\theta\cos\theta\ ,\ 2a\cos\theta - a\theta\sin\theta\ ,\ 0\rangle\ .
\end{aligned}$$

And thus

$$\begin{aligned}
\mathbf{r}'(\theta)\times\mathbf{r}''(\theta) &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k}\\ a\cos\theta - a\theta\sin\theta & a\sin\theta + a\theta\cos\theta & b\\ -2a\sin\theta - a\theta\cos\theta & 2a\cos\theta - a\theta\sin\theta & 0\end{vmatrix}\\
&= \langle ab\theta\sin\theta - 2ab\cos\theta\ ,\ -ab\theta\cos\theta - 2ab\sin\theta\ ,\\
&\qquad a^2\big((\cos\theta - \theta\sin\theta)(2\cos\theta - \theta\sin\theta) + (2\sin\theta + \theta\cos\theta)(\sin\theta + \theta\cos\theta)\big)\rangle\\
&= \left\langle ab\theta\sin\theta - 2ab\cos\theta\ ,\ -ab\theta\cos\theta - 2ab\sin\theta\ ,\ 2a^2 + a^2\theta^2\right\rangle\ ,\\
\left|\mathbf{r}'(\theta)\times\mathbf{r}''(\theta)\right| &= \sqrt{(ab\theta\sin\theta - 2ab\cos\theta)^2 + (-ab\theta\cos\theta - 2ab\sin\theta)^2 + (2a^2 + a^2\theta^2)^2}\\
&= a\sqrt{b^2\theta^2 + 4b^2 + 4a^2 + 4a^2\theta^2 + a^2\theta^4} = a\sqrt{a^2\theta^4 + (3a^2+1)\,\theta^2 + 4}
\end{aligned}$$

As a result,

$$\kappa(\theta) = \frac{\left|\mathbf{r}'(\theta)\times\mathbf{r}''(\theta)\right|}{\left|\mathbf{r}'(\theta)\right|^3} = \frac{a\sqrt{a^2\theta^4 + (3a^2+1)\,\theta^2 + 4}}{(a^2\theta^2+1)^{\frac{3}{2}}} = \frac{a\sqrt{\theta^2 + 3 + \frac{1}{a^2\theta^2+1}}}{a^2\theta^2+1}\ .$$

(correct use of formulas: 3 points)
(correct calculations: 3 points)

5. Let $f(x, y) = \begin{cases} \dfrac{x^3}{\ln(1 + x^2 + y^2)} & \text{if } (x, y) \neq (0, 0)\ ,\\ 0 & \text{if } (x, y) = (0, 0)\ .\end{cases}$

(a) Show that f is continuous at $(0, 0)$. (3 points)
(Hint: you may use polar coordinates (r, θ) with $r \geq 0$ to represent the point (x, y) and notice that $r \to 0^+$ when $(x, y) \to (0, 0)$.)

(b) Find the gradient vector $\nabla f(0, 0)$. (3 points)

Cont.

(c) Determine whether f_y is continuous at $(0,0)$. (3 points)

(d) By computing the directional derivative $D_{\mathbf{u}}f(0,0)$ at $(0,0)$ in the direction of an arbitrary unit vector $\mathbf{u}$, determine whether f is differentiable at $(0,0)$. (4 points)

Solution:

(a) Putting $x = r\cos\theta$ and $y = r\sin\theta$ with $r > 0$, one has

$$|f(r\cos\theta, r\sin\theta)| = \frac{r^3\left|\cos^3\theta\right|}{\ln(1+r^2)} \leq r\,\frac{r^2}{\ln(1+r^2)}\,.$$

Since the right-hand-side converges to $0\cdot\frac{1}{\ln e} = 0$ as $r \to 0^+$, it follows from the Squeeze Theorem that

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{r\to 0^+} f(r\cos\theta, r\sin\theta) = 0 = f(0,0)\,,$$

which means that f is continuous at $(0,0)$. (correct argument: 3 points)

(b) Since $f(0,y) = 0$ for all $y \in \mathbb{R}$, we have $f_y(0,y) = 0$ for all $y \in \mathbb{R}$, and $f_y(0,0) = 0$ in particular.

A direct computation shows that

$$f_x(0,0) = \lim_{h\to 0}\frac{f(h,0)-f(0,0)}{h} = \lim_{h\to 0}\frac{h^2}{\ln(1+h^2)} = \frac{1}{\ln e} = 1\,.$$

As a result, $\nabla f(0,0) = \langle 1, 0\rangle$. (calculation + answer: 2+1 points)

(c) For arbitrary $(x,y) \neq (0,0)$, we have

$$f_y(x,y) = \frac{\partial}{\partial y}\left(\frac{x^3}{\ln(1+x^2+y^2)}\right) = -\frac{2x^3 y}{(1+x^2+y^2)(\ln(1+x^2+y^2))^2}\,.$$

Restricting f_y to the line $x = y$, we have

$$f_y(x,x) = -\frac{2x^4}{(1+2x^2)(\ln(1+2x^2))^2} \to -\frac{2}{4(\ln e)^2} = -\frac{1}{2} \quad \text{as} \quad x \to 0\,.$$

However, $f_y(0,0) = 0$ by previous calculation. Therefore, f_y is *not* continuous at $(0,0)$. (calculation + conclusion: 2+1 points)

(d) Take an arbitrary unit vector $\mathbf{u} = \langle\cos\theta\,,\,\sin\theta\rangle$. By definition,

$$D_{\mathbf{u}}f(0,0) = \lim_{h\to 0}\frac{f(h\cos\theta, h\sin\theta) - f(0,0)}{h} = \lim_{h\to 0} h^2\frac{\cos^3\theta}{\ln(1+h^2)} = \cos^3\theta\,.$$

Note also that $\nabla f(0,0)\cdot\mathbf{u} = \cos\theta$ by previous calculation. As a result, when $\cos\theta \neq 0$ and $\cos^2\theta \neq 1$, for example, when $\theta = \frac{\pi}{4}$, $D_{\mathbf{u}}f(0,0)$ and $\nabla f(0,0)\cdot\mathbf{u}$ do *not coincide*. This implies that f is not differentiable at $(0,0)$. (argument + conclusion: 1+1 points)

(calculation of $D_{\mathbf{u}}f(0,0)$ for a proper choice of $\mathbf{u}$: 2 points)

6. Let $f(u,v)$ be a differentiable function on $(0,\infty)\times(0,\infty)$ and consider the change of coordinates

$$u = x + y\,, \quad v = \frac{y}{x+y}$$

for any $(x,y) \in (0,\infty)\times(0,\infty)$. As usual, let $\mathbf{i}$ and $\mathbf{j}$ be the standard basic unit vectors in the directions of the positive x- and y-axes respectively. Also write $f_u := \frac{\partial f}{\partial u}$ and $f_v := \frac{\partial f}{\partial v}$.

(a) Treat f as a function on the (x, y)-plane and find its gradient ∇f in terms of $\mathbf{i}$, $\mathbf{j}$, x, y, f_u and f_v. (4 points)

(b) Let $\mathbf{e}_u$ and $\mathbf{e}_v$ be respectively the *unit* vectors in the directions where u and v increase at the fastest rates with respect to the changes of x and y. Show that

$$\nabla f = \sqrt{2} f_u \mathbf{e}_u + \frac{\sqrt{x^2+y^2}}{(x+y)^2} f_v \mathbf{e}_v \,. \qquad \text{(4 points)}$$

(c) Find the directional derivatives $D_{\mathbf{e}_u} f$ and $D_{\mathbf{e}_v} f$. (4 points)
(Caution: $\mathbf{e}_u$ and $\mathbf{e}_v$ are *not* orthogonal.)

Solution:

(a) Note that

$$\begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -\frac{y}{(x+y)^2} & \frac{x}{(x+y)^2} \end{bmatrix} .$$

The Chain Rule yields

$$\begin{aligned} \frac{\partial f}{\partial x} &= f_u \frac{\partial u}{\partial x} + f_v \frac{\partial v}{\partial x} = f_u - f_v \frac{y}{(x+y)^2} \,, \\ \frac{\partial f}{\partial y} &= f_u \frac{\partial u}{\partial y} + f_v \frac{\partial v}{\partial y} = f_u + f_v \frac{x}{(x+y)^2} \,. \end{aligned} \qquad \text{(3 points)}$$

Therefore,

$$\nabla f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} = \left(f_u - f_v \frac{y}{(x+y)^2} \right) \mathbf{i} + \left(f_u + f_v \frac{x}{(x+y)^2} \right) \mathbf{j} \,. \qquad \text{(1 point)}$$

(b) It follows from the description of $\mathbf{e}_u$ and $\mathbf{e}_v$ that

$$\begin{aligned} \mathbf{e}_u &= \frac{\nabla u}{|\nabla u|} = \frac{\frac{\partial u}{\partial x}\mathbf{i} + \frac{\partial u}{\partial y}\mathbf{j}}{\sqrt{\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2}} = \frac{1}{\sqrt{2}}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{j} \,, \\ \mathbf{e}_v &= \frac{\nabla v}{|\nabla v|} = \frac{-y\mathbf{i} + x\mathbf{j}}{(x+y)^2 \sqrt{\left(\frac{-y}{(x+y)^2}\right)^2 + \left(\frac{x}{(x+y)^2}\right)^2}} = -\frac{y}{\sqrt{x^2+y^2}}\mathbf{i} + \frac{x}{\sqrt{x^2+y^2}}\mathbf{j} \,. \end{aligned} \qquad \text{(3 points)}$$

Therefore,

$$\sqrt{2} f_u \mathbf{e}_u + \frac{\sqrt{x^2+y^2}}{(x+y)^2} f_v \mathbf{e}_v = f_u(\mathbf{i}+\mathbf{j}) + f_v \left(\frac{-y}{(x+y)^2}\mathbf{i} + \frac{x}{(x+y)^2}\mathbf{j} \right) = \nabla f \,. \qquad \text{(1 point)}$$

(c) It follows from direct calculations that

$$\begin{aligned} D_{\mathbf{e}_u} f = \nabla f \cdot \mathbf{e}_u &= \frac{1}{\sqrt{2}} \left(f_u - f_v \frac{y}{(x+y)^2} \right) + \frac{1}{\sqrt{2}} \left(f_u + f_v \frac{x}{(x+y)^2} \right) \qquad \text{(2 points)} \\ &= \sqrt{2} f_u + \frac{x-y}{\sqrt{2}(x+y)^2} f_v \,, \\ D_{\mathbf{e}_v} f = \nabla f \cdot \mathbf{e}_v &= \frac{-y}{\sqrt{x^2+y^2}} \left(f_u - f_v \frac{y}{(x+y)^2} \right) + \frac{x}{\sqrt{x^2+y^2}} \left(f_u + f_v \frac{x}{(x+y)^2} \right) \\ &= \frac{x-y}{\sqrt{x^2+y^2}} f_u + \frac{\sqrt{x^2+y^2}}{(x+y)^2} f_v \,. \end{aligned}$$

(2 points)

7. Let $f(x,y) := -y^3 + x^2 + y^2 - xy$.

   (a) Find all the local maximum and minimum values of f and locate its saddle points, if any. (6 points)

   (b) Find the extreme values of f on the region bounded by the square with vertices $(1,1)$, $(1,-1)$, $(-1,1)$ and $(-1,-1)$. (6 points)

Solution:

(a) To look for critical points of f in $\mathbb{R}^2$, notice that

$$\nabla f = \big\langle 2x - y\,,\ -3y^2 + 2y - x \big\rangle = 0 \iff \begin{cases} x = \dfrac{y}{2} \\ 2y^2 - y = 0 \end{cases}$$
$$\iff (x,y) = (0,0) \quad \text{or} \quad \left(\frac{1}{4}, \frac{1}{2}\right)\,,$$

that is, $(0,0)$ and $\left(\frac{1}{4}, \frac{1}{2}\right)$ are the only critical points of f on the whole plane. (3 points)
Note also that

$$f_{xx} = 2 > 0 \quad \text{and} \quad D(x,y) := \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = \begin{vmatrix} 2 & -1 \\ -1 & -6y+2 \end{vmatrix} = -12y + 3\,,$$

and thus $D(0,0) = 3 > 0$ and $D\left(\frac{1}{4}, \frac{1}{2}\right) = -3 < 0$. The Second Derivatives Test asserts that $\left(\frac{1}{4}, \frac{1}{2}\right)$ is a saddle point of f and $f(0,0) = 0$ is a local minimum value around the critical point $(0,0)$. (3 points)

(b) Notice that the local minimal point $(0,0)$ is contained in the given region. To look for extreme values of f on the boundary of the given square, one searches for the critical points of f when restricted to the lines $x = \pm 1$ and $y = \pm 1$ and obtains

$$f_x(x,1) = 2x - 1 = 0 \quad \Rightarrow \quad x = \frac{1}{2} \quad \text{and} \quad f\left(\frac{1}{2}, 1\right) = -\frac{1}{4}\,,$$

$$f_x(x,-1) = 2x + 1 = 0 \quad \Rightarrow \quad x = -\frac{1}{2} \quad \text{and} \quad f\left(-\frac{1}{2}, -1\right) = \frac{7}{4}\,,$$

$$\begin{aligned} f_y(-1,y) &= -3y^2 + 2y + 1 \\ &= -(3y+1)(y-1) = 0 \end{aligned} \quad \Rightarrow \quad y = -\frac{1}{3} \ \text{or} \ 1 \quad \text{and} \quad \begin{cases} f\left(-1, -\dfrac{1}{3}\right) = \dfrac{22}{27} \\ f(-1,1) = 2\,, \end{cases}$$

$$f_y(1,y) = -3y^2 + 2y - 1 = -3\left(y - \frac{1}{3}\right)^2 - \frac{2}{3} < 0 \quad \Rightarrow \quad \text{no critical point on the line } x = 1.$$

Moreover, one has

$$f(1,1) = 0\,, \quad f(1,-1) = 4 \quad \text{and} \quad f(-1,-1) = 2\,.$$

(computing values at the 7 critical points on the boundary of the square: 4 points)
Therefore, comparing the values of f at the points $(0,0)$, $(1,1)$, $(1,-1)$, $(-1,1)$, $(-1,-1)$, $\left(\frac{1}{2}, 1\right)$, $\left(-\frac{1}{2}, -1\right)$ and $\left(-1, -\frac{1}{3}\right)$, one can conclude that f attains the absolute minimum value $f\left(\frac{1}{2}, 1\right) = -\frac{1}{4}$ and the absolute maximum value $f(1,-1) = 4$ on the given region.
(comparing with $f(0,0)$: 1 point)
(correct extreme values: 1 point)

8. Find the point closest to the origin on the curve of intersection of the plane $x + y + z = 1$ and the cone $z^2 = 2x^2 + 2y^2$. (10 points)

(Suggestion: if you are using Lagrange's method to solve this problem, you are recommended to start by showing that the Lagrange multipliers cannot be some special values.)

Solution: Let $f(x, y, z) := x^2 + y^2 + z^2$. The aim is to minimize f under the constraints $x + y + z = 1$ and $z^2 = 2x^2 + 2y^2$. Write $g(x, y, z) := x + y + z$ and $h(x, y, z) := 2x^2 + 2y^2 - z^2$ for convenience.

By the method of Lagrange multipliers, one obtains

$$\begin{cases} \nabla f = \lambda \nabla g + \mu \nabla h \\ g(x, y, z) = 1 \\ h(x, y, z) = 0 \end{cases} \iff \begin{cases} 2(1 - 2\mu)x = \lambda \\ 2(1 - 2\mu)y = \lambda \\ 2(1 + \mu)z = \lambda \\ x + y + z = 1 \\ 2x^2 + 2y^2 - z^2 = 0 \end{cases}.$$

(correct use of Lagrange's method: 3 points)

(correct equations: 2 points)

First note that $\lambda = 0$ is inconsistent with the system, since, if $(1-2\mu) \neq 0$, the first two equations would imply $(x, y) = (0, 0)$, which is inconsistent with the last two equations; if $(1 - 2\mu) = 0$, the third equation would force z to be 0, which is also inconsistent with the last two equations. Therefore, one must have $\lambda \neq 0$.

It follows from the first two equations that $(1 - 2\mu) \neq 0$ and thus $x = y$. Substituting this into the last two equations yields

$$\begin{cases} x = y \\ 2x + z = 1 \\ 4x^2 - z^2 = 0 \end{cases} \iff (x, y, z) = \left(\frac{1}{4}, \frac{1}{4}, \frac{1}{2}\right).$$

(process of solving the system + answer: 3+2 points)

It can be seen easily that the curve of intersection of the given plane and cone (which is indeed a parabola) contains points arbitrarily far away from the origin, so f does not have a maximum under the given constraints. Therefore, the point $\left(\frac{1}{4}, \frac{1}{4}, \frac{1}{2}\right)$ obtained above has to be the minimal point of f, i.e. the point on the given curve of intersection which is the closest to the origin.

The End.