微乙小考二 (2018/3/29)

1. (5%) 求出 $f(x,y)=\sin(xy)+\cos(x^2y)$ 在點 $(1,\frac{\pi}{4})$ 沿著向量 $\vec{u}=(1,3)$ 的方向導數。

sol: $\nabla f(x,y)=\Big(\cos(xy)\cdot y-\sin(x^2y)\cdot 2xy, \cos(xy)\cdot x-\sin(x^2y)\cdot x^2\Big)$

$\nabla f(1,\frac{\pi}{4})=\Big(\frac{\sqrt{2}}{2}\cdot\frac{\pi}{4}-\frac{\sqrt{2}}{2}\cdot\frac{\pi}{2}, \frac{\sqrt{2}}{2}-\frac{\sqrt{2}}{2}\Big)=\Big(-\frac{\sqrt{2}}{8}\pi,0\Big)$

Take the unit vector of $\vec{u}\Rightarrow\frac{\vec{u}}{|\vec{u}|}=\Big(\frac{1}{\sqrt{10}},\frac{3}{\sqrt{10}}\Big)$

$\frac{\partial f}{\partial\vec{u}}(1,\frac{\pi}{4})=\Big(-\frac{\sqrt{2}}{8}\pi,0\Big)\cdot\Big(\frac{1}{\sqrt{10}},\frac{3}{\sqrt{10}}\Big)=\frac{-\pi}{8\sqrt{5}}$

2. (5%) 求曲面 $xyz^2=6$ 在點 $(3,2,1)$ 的切平面方程式。

sol: The tangent plane Γ at (3,2,1) can be derived from

$$f_x(3,2,1)(x-3)+f_y(3,2,1)(y-2)+f_z(3,2,1)(z-1)=0$$

Let $f(x,y,z)=xyz^2-6=0$

$$f_x=\frac{\partial f}{\partial x}(x,y,z)=yz^2 \qquad f_x(3,2,1)=2$$

$$f_y=\frac{\partial f}{\partial y}(x,y,z)=xz^2 \qquad f_y(3,2,1)=3$$

$$f_z=\frac{\partial f}{\partial z}(x,y,z)=2xyz \qquad f_z(3,2,1)=12$$

Therefore, we have $\Gamma: 2(x-3)+3(y-2)+12(z-1)=0$. Namely, $2x+3y+12z=24$.

3. (10%) 找出函數 $f(x,y)=e^y(y^2-x^2)$ 的候選點，並決定它是局部極大值，局部極小值或是鞍點。

sol:

$$\nabla f(x,y)=0\Rightarrow\begin{cases} f_x(x,y)=-2xe^y=0 & (1)\\ f_y(x,y)=e^y(y^2-x^2)+2ye^y=0 & (2)\end{cases}$$

Solving (1) and (2), we do have $y(y+2)=0$. There are two critical points $(0,0)$ and $(0,-2)$

Check these points further by calculate $D(x)=\begin{vmatrix} f_{xx} & f_{xy}\\ f_{yx} & f_{yy}\end{vmatrix}=\begin{vmatrix} -2e^y & (-2x)e^y\\ (-2x)e^y & e^y(y^2+4y+2-x^2)\end{vmatrix}$

* For $(0,0)$
  $D(0,0)=\begin{vmatrix} -2 & 0\\ 0 & 2\end{vmatrix}=-4<0$. Therefore, (0,0) is a saddle point.
* For $(0,-2)$
  $D(0,-2)=\begin{vmatrix} -2e^{-2} & 0\\ 0 & -2e^{-2}\end{vmatrix}=4e^{-4}>0$ and $\frac{\partial^2 f}{\partial x^2}\Big|_{(0.-2)}=-2e^{-2}<0$.
  Hence $(0,-2)$ is a local maximum.