1. (12%) 計算瑕積分 $\int_{0^+}^{\infty} \frac{e^{-3x}}{\sqrt{2x}} dx$ 之值。

Solution:

令 $u = \sqrt{3}\sqrt{x},\ \ du = \frac{\sqrt{3}}{2}\frac{1}{\sqrt{x}}dx$...(4 pt)

$$原 = \frac{1}{\sqrt{2}}\int_0^{\infty} \frac{2}{\sqrt{3}} e^{-u^2} du = \sqrt{\frac{2}{3}}\int_0^{\infty} e^{-u^2} du = \frac{\sqrt{2}}{\sqrt{3}}\frac{\sqrt{\pi}}{2} = \frac{\sqrt{\pi}}{\sqrt{6}}$$ ...(3+3+2 pt)

2. (12%) 假設一電力系統故障頻率遵守 Poisson 分配，且平均每 15 天故障 7 次。求在某一天中該電力系統故障不超過一次的機率。

Solution:

Since it's Poisson distribution, we have

$$P(k) = \frac{(\lambda T)^k}{k!} e^{-\lambda T} \tag{1}$$

where $\lambda = \frac{7}{15}$ and $T = 1$ So

$$P(k) = \frac{(\frac{7}{15})^k}{k!} e^{-\frac{7}{15}} \tag{2}$$

Hence the answer is

$$P(0) + P(1) = \frac{7}{15} e^{-\frac{7}{15}} + e^{-\frac{7}{15}} = \frac{22}{15} e^{-\frac{7}{15}} \tag{3}$$

GRADING CRITERIA

Write down equation (1), (2) and (3) correctly get 4 points respectively.

3. (13%) 令兩隨機變數 X, Y 獨立，其機率密度函數為

$$f_X(t) = te^{-t}, \quad f_Y(t) = \frac{1}{2}t^2e^{-t} \text{當 } t \geq 0.$$

當 $t < 0$ 時， $f_X(t)$, $f_Y(t)$ 均為 0。令隨機變數 $Z = X + Y$，求 Z 的機率密度函數 $f_Z(t)$。

Solution:

(1分)由於X, Y只有在正數$(t > 0)$取值$(f_X, f_Y > 0)$, 所以$Z = X + Y$也只會在正數取值, 故$f_Z(t)$取值範圍是$t > 0$.
ps:寫$t < 0, f_Z(t) = 0$, 或$t \geq 0$, 或都寫都可以.

(4分)如果$Z = t, Y = s$則$X = Z - Y = t - s$, 所以我們有$f_Z(t) = \int_{-\infty}^{\infty} f_X(t-s)f_Y(s)ds$,

(4分)$= \int_0^t f_X(t-s)f_Y(s)ds$ 因為f_X, f_Y只在正數取值, 故在$s > 0$且$t - s > 0$(即$0 < s < t$)下的部分才有值,

(6分)$= \int_0^t (t-s)e^{-(t-s)}\frac{1}{2}s^2e^{-s}ds = \frac{e^{-t}}{2}\int_0^t s^2t - s^3 ds = \frac{t^4e^{-t}}{24}$.
ps:範圍錯誤,計算錯誤每處斟酌扣1到2分.

4. (15%) X 與 Y 為兩互相獨立之隨機變數，X取值在$\{-1,1\}$，Y取值在$\{-1,1\}$ 。若已知道聯合機率函數

$$\begin{aligned} P(X=1,Y=-1) &= \frac{8}{15} \\ P(X=-1,Y=-1) &= \frac{4}{15} \\ P(X=1,Y=1) &= \frac{2}{15} \\ P(X=-1,Y=1) &= \frac{1}{15} \end{aligned}$$

令 $Z = \dfrac{X\cdot Y+1}{2}$。

(a) (3%) 求 Z 之值域及其機率函數。

(b) (3% each, 6 %total) 求 $\mathrm{E}(Z)$ 與 $\mathrm{Var}(Z)$。

(c) (2% each, 6 %total) 若隨機變數 $W_n \sim Z$ 並且互相獨立，令

$$W = W_1 + W_2 + \cdots + W_{10}.$$

求 $\mathrm{P}(W=4)$、期望值 $\mathrm{E}(W)$ 及變異數 $\mathrm{Var}(W)$。

Solution:

(a) Z 之值域為 $\{1,0\}$ 且

$$\begin{aligned} P(Z=1) = P(X=1,Y=1) + P(X=-1,Y=-1) = \frac{2}{5}, \\ P(Z=0) = P(X=1,Y=-1) + P(X=-1,Y=1) = \frac{3}{5}. \end{aligned}$$

(b) 由 (a) 知 $Z \sim B\left(1, \frac{2}{5}, \frac{3}{5}\right)$, 因此 $\mathrm{E}(Z) = 1\cdot\frac{2}{5} = \frac{2}{5}$, $\mathrm{Var}(Z) = 1\cdot\frac{2}{5}\cdot\frac{3}{5} = \frac{6}{25}$.

(c) 由獨立性得 $W \sim B\left(10, \frac{2}{5}, \frac{3}{5}\right)$, 因此

$$P(W=4) = C_4^{10}\left(\frac{2}{5}\right)^4\left(\frac{3}{5}\right)^6,\ \mathrm{E}(W) = 10\cdot\frac{2}{5} = 4,\ \mathrm{Var}(Z) = 10\cdot\frac{2}{5}\cdot\frac{3}{5} = \frac{12}{5}.$$

(a) 值域1分，機率函數2分 (b) 列式正確1分，答案正確2分 (c) 列式正確1分，答案正確1分

5. (12%) 假設到速食店用餐，設隨機變數 W 表示等候時間。設 W 之平均值為 2 分鐘且為指數分佈。

(a) (3%) 寫下 W 之機率密度函數。

(b) (3%) 求在 1 到 3分鐘間的機率 $\mathrm{P}(1 < W < 3)$ 為何?

(c) (3% each, 6% total) 求期望值 $\mathrm{E}(W)$ 及 變異數 $\mathrm{Var}(W)$，請列計算過程。

Solution:

(a.)

$f_w(t) = \lambda e^{\lambda t}, \lambda = \frac{1}{2}$

(b.)

$P(1 < w < 3) = \int_1^3 \frac{1}{2} e^{-\frac{t}{2}}\ dt = -(2)e^{-\frac{t}{2}}|_1^3 = e^{-\frac{1}{2}} - e^{-\frac{3}{2}}$

(c.)

$E(w) = \int_0^\infty t f_w(t)\ dt = \int_0^\infty \frac{t}{2} e^{-\frac{t}{2}}\ dt$

$(assume u = t, v' = e^{-\frac{t}{2}}, \int uv' = uv - \int u'v)$

$= \frac{1}{2}((-2)te^{-\frac{t}{2}}|_0^\infty - \int_0^\infty (-2)e^{-\frac{t}{2}}\ dt)$

$= 2$

$E(w^2) = \int_0^\infty t^2 f_w(t)\ dt = \int_0^\infty \frac{t^2}{2} e^{-\frac{t}{2}}\ dt$

$(assume u = t^2, v' = e^{-\frac{t}{2}}, \int uv' = uv - \int u'v)$

$= \frac{1}{2}((-2)t^2 e^{-\frac{t}{2}}|_0^\infty - \int_0^\infty (-2)(2t)e^{-\frac{t}{2}}\ dt)$

$= \frac{1}{2}((-2)t^2 e^{-\frac{t}{2}}|_0^\infty + 4((-2)te^{-\frac{t}{2}}|_0^\infty - \int_0^\infty (-2)e^{-\frac{t}{2}}\ dt))$

$= 8$

$Var(w) = E(w^2) - (E(w))^2 = 4$

score: (a) if answer is correct get 3 points, or 0 point. However, writing $f_w(t) = \lambda e^{\lambda t}$ get one point.

(b.) if writing the definition or only answer get one point

(c.) if writing the definition or only answer get one point. Only write $E(w) = \frac{1}{\lambda}, Var(w) = \frac{1}{\lambda^2}$ get one point separately.

6. (12%) 解微分方程式 $(t^2+2)y'(t)+(4t)y(t)=1$ 且滿足初始值條件 $y(0)=2$.

Solution:

$$原 \iff y'+(\frac{4t}{t^2+2})y=\frac{1}{t^2+2} \quad (式一) \ ...(1 \ \ pt)$$

Let $p(t)=\dfrac{4t}{t^2+2}$

$$\int p(t)dt=\int \frac{4t}{t^2+2}dt=2\ln(t^2+2)+C_0, \ 取 \ \ C_0=0 \ ...(2 \ \ pt)$$

將(式一) 兩邊同乘以

$$e^{\int p(t)dt}=e^{2\ln(t^2+2)}=(t^2+2)^2 \ ...(1 \ \ pt)$$

得到

$$原 \iff (t^2+2)^2y'+(4t)(t^2+2)y=t^2+2 \quad \iff ((t^2+2)^2\cdot y)'=t^2+2 \ \ ...(2 \ \ pt)$$

故

$$(t^2+2)^2\cdot y=\int (t^2+2)dt=\frac{t^3}{3}+2t+C \ \ ...(2 \ \ pt)$$

即

$$y=\frac{\frac{t^3}{3}+2t+C}{(t^2+2)^2}$$

但是，由初始條件: $y(0)=2$, 當 $t=0$ 時 $y=2$

$$y(0)=2=\frac{C}{4} \ \Rightarrow \ C=8 \ \ ...(2 \ \ pt)$$

所以我們有

$$y=\frac{\frac{t^3}{3}+2t+8}{(t^2+2)^2} \ \ ...(2 \ \ pt)$$

7. (12%) 解微分方程式 $y'(t) = \sin t + \sin t(y(t))^2$. 求一般解。

Solution:

This differential equation is separable

$$y'(t) = \sin t\ (1 + y^2(t)) \tag{1}$$

$$\frac{y'(t)}{1 + y^2(t)} = \sin t \tag{2}$$

Now integrate both side of equation

$$\int \frac{y'(t)}{1 + y^2(t)}\ dt = \int \sin t\ dt \tag{3}$$

$$\int \frac{dy}{1 + y^2} = \int \sin t\ dt \tag{4}$$

$$\tan^{-1} y = -\cos t + C \tag{5}$$

Hence the general solution of this differential is

$$y = \tan(\cos t + C) \tag{6}$$

where C is a constant.

GRADING CRITERIA

2 points for each of the equation above. Note that if you try to solve this equation by finding integrating factor, you will get 2 points only.

8. (12%) 若連續型隨機變數 X 所對應的機率密度函數為 $f_X(t) = \frac{1}{\sqrt{\pi}}e^{-t^2}$。令隨機變數 $W = X^3$，求 W 所對應之機率密度函數。

Solution:

用分配函數(c.d.f.)法:

(4分)$F_W(t) = P(W < t) = P(X^3 < t) = P(X < t^{\frac{1}{3}}) = \int_{-\infty}^{t^{\frac{1}{3}}} f_X(t)$.

(2分)因為$f_W(t) = \frac{d}{dt}F_W(t)$,

(2分)W的機率密度函數是 $f_W(t) = \frac{d}{dt}(\int_{-\infty}^{t^{\frac{1}{3}}} f_X(s)ds)$

(4分)$= \frac{d}{dt^{\frac{1}{3}}}(\int_{-\infty}^{t^{\frac{1}{3}}} f_X(s)ds)\frac{dt^{\frac{1}{3}}}{dt} = f_X(t^{\frac{1}{3}}) \cdot \frac{1}{3}t^{-\frac{2}{3}} = \frac{t^{-\frac{2}{3}}e^{-t^{\frac{2}{3}}}}{3\sqrt{\pi}}$.

使用了嵌入法跟微積分基本定理.

ps:細節會依個人寫法差異做些微配分調整.