1. (20 points) Evaluate the integrals.

   (a) (10 points) $\int_0^1 \int_{\sqrt{y}}^{2\sqrt{y}} e^{x^3}\, dx\, dy + \int_1^4 \int_{\sqrt{y}}^{2} e^{x^3}\, dx\, dy.$

   (b) (10 points) $\iint_R \cos\left(\frac{y-2x}{y+x}\right) dA$, where R is the trapezoidal region with vertices (1,0),(2,0),(0,2) and (0,1).

Solution:

(a) Let D_1, D_2 be the region such that

$$\int\int_{D_1} e^{x^3} dA = \int_0^1 \int_{\sqrt{y}}^{2\sqrt{y}} e^{x^3} dxdy$$

,

$$\int\int_{D_2} e^{x^3} dA = \int_1^4 \int_{\sqrt{y}}^{2e^{x^3}} dxdy$$

. Therefore, $\int_1^4 \int_{\sqrt{y}}^{2e^{x^3}} dxdy + \int_0^1 \int_{\sqrt{y}}^{2\sqrt{y}} e^{x^3} dxdy = \int\int_{D_1 \cup D_2} e^{x^3} dA = \int_0^2 \int_{\frac{x^2}{4}}^{x^2} e^{x^3} dydx = \int_0^2 \frac{3}{4} x^2 e^{x^3} dx = \frac{1}{4}(e^8 - 1).$

standard of evalutaion

Simple calculation error 8pt
Right integration range after changing the order of x and y 5pt
Right integration range after changing the order of x and y but wrong calculation 4pt

(b) Let $u = x + y,\ v = y - 2x$, then we have $x = \frac{1}{3}(u - v),\ y = \frac{1}{3}(v + 2u)$. (1 point)

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{1}{3} & \frac{-1}{3} \\ \frac{2}{3} & \frac{1}{3} \end{vmatrix} = \frac{1}{3} \qquad \text{(2 points)}$$

Boundary: $v = u,\ v = -2u,\ u = 1$ and $u = 2$. (2 points)

$$\iint_R \cos(\frac{y-2x}{x+y})dA = \iint_D \cos(\frac{v}{u}) \mid \frac{\partial(x,y)}{\partial(u,v)} \mid dA \qquad \text{(2 points)}$$
$$= \iint_D \cos(\frac{v}{u})\frac{1}{3}dA$$
$$= \frac{1}{3}\int_1^2 \int_{-2u}^{u} \cos(\frac{v}{u})dvdu$$
$$= \frac{1}{3}\int_1^2 (u\sin(\frac{v}{u}) \mid_{v=-2u}^{v=u})du$$
$$= \frac{1}{3}\int_1^2 u(\sin 1 + \sin 2)du$$
$$= \frac{1}{2}(\sin 1 + \sin 2) \qquad \text{(3 points)}$$

2. (16 points) Evaluate the integrals.

(a) (8 points) $\int_0^2 \int_0^1 \int_y^1 e^{-z^2}\, dz\, dy\, dx$.

(b) (8 points) $\iiint_E x^2 dV$, where E is the solid that lies in the first octant within the cylinder $x^2+y^2=1$ and below the cone $z^2 = 4x^2 + 4y^2$.

Solution:

(a) $\int_0^2 \int_0^1 \int_y^1 e^{-z^2} dzdydx$

$= \int_0^2 \int_0^1 \int_z^0 e^{-z^2} dydzdx$ (3 pt)

$= \int_0^2 \int_0^1 ze^{-z^2} dzdx$, let $u = -z^2 du = -2zdz$ (3pt)

$= \int_0^2 \int_0^{-1} e^u dudx$

$= \int_0^2 -\frac{1}{2}e^{-1} + \frac{1}{2} dx$ (2pt)

$= 1 - e^{-1}$

(b) (Method 1) Use cylindrical coordinates

$$\iiint_E x^2\, dV = \overbrace{\int_0^{\pi/2} \int_0^1 \int_0^{2r} (r\cos\theta)^2 r\, dz\, dr\, d\theta}^{4\text{pts}} = \overbrace{\int_0^{\pi/2} \int_0^1 2r^4 \cos^2\theta\, dr\, d\theta}^{1\text{pt}} = \overbrace{\frac{\pi}{10}}^{3\text{pts}}$$

or

$$\iiint_E x^2\, dV = \int_0^{\pi/2} \int_0^2 \int_{z/2}^1 (r\cos\theta)^2 r\, dr\, dz\, d\theta = \cdots = \frac{\pi}{10}$$

(Method 2) Use spherical coordinates

$$\iiint_E x^2\, dV = \overbrace{\int_0^{\pi/2} \int_{\tan^{-1}\frac{1}{2}}^{\pi/2} \int_0^{\frac{1}{\sin\phi}} (r\sin\phi\cos\theta)^2 r^2 \sin\phi\, dr\, d\phi\, d\theta}^{4\text{pts}} = \overbrace{\frac{1}{5}\int_0^{\pi/2} \int_{\tan^{-1}\frac{1}{2}}^{\pi/2} \frac{\cos^2\theta}{\sin^2\phi}\, d\phi\, d\theta}^{1\text{pt}}$$

$$= \frac{1}{5}\int_0^{\pi/2} \cos^2\theta\, d\theta \int_{\tan^{-1}\frac{1}{2}}^{\pi/2} \csc^2\phi\, d\phi = \frac{1}{5}\cdot\frac{\pi}{4}\cdot 2 = \frac{\pi}{10} \quad \text{(3pts)},$$

where $\int \csc^2\phi\, d\phi = -\cot\phi + C$

(Method 3) By symmetry and use cylindrical coordinates

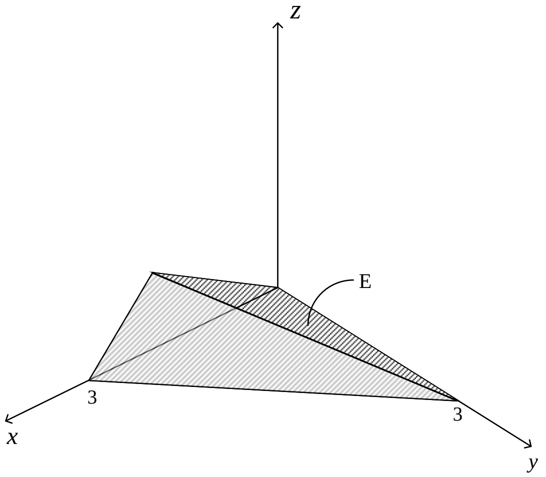
$$\iiint_E x^2\, dV = \frac{1}{2}\iiint_E x^2 + y^2\, dV = \frac{1}{2}\int_0^{\pi/2} \int_0^1 \int_0^{2r} r^2 \cdot r\, dz\, dr\, d\theta = \frac{\pi}{10}$$

Remark 常見錯誤與給分

θ範圍寫成$\int_0^{2\pi}$或$\int_0^{\pi/4}$算出答案$\frac{2\pi}{5}$或$\frac{\pi+2}{20}$得6分。E算成椎體內部得到$\frac{\pi}{40}$得5分。

z範圍寫成$\int_0^{4r^2}$得到$\frac{\pi}{6}$或漏寫Jacobian r得4分。z範圍寫成$\int_0^2$得到$\frac{\pi}{8}$得3分。

3. (10 points) Let E be the tetrahedron bounded by the planes $x + y + z = 3$, $x = 2z$, $y = 0$, and $z = 0$ which is completely occupied by a solid with the density function $\rho(x, y, z) = y$. Find the total mass of the solid.



Solution:

The total mass of the solid is given by $M := \iiint_E y \, dV$. (3 points)

The region E can be described as

$$E = \left\{ (x, y, z) \in \mathbb{R}^3 \,\middle|\, (z, x) \in D \, , \, 0 \le y \le 3 - z - x \right\} \, ,$$

where D is the triangular region in the (z, x)-plane given by

$$D := \left\{ (z, x) \in \mathbb{R}^2 \,\middle|\, 0 \le z \le 1 \, , \, 2z \le x \le 3 - z \right\} \, .$$

(correct description of the region (reflected in the regions of integration): 4 points)

[Method 1] By a direct computation, the total mass is found to be

$$\begin{aligned} M &= \iiint_E y \, dV = \int_0^1 \int_{2z}^{3-z} \int_0^{3-z-x} y \, dy \, dx \, dz \\ &= \int_0^1 \int_{2z}^{3-z} \left[\frac{y^2}{2} \right]_{y=0}^{y=3-z-x} dx \, dz = \frac{1}{2} \int_0^1 \int_{2z}^{3-z} (3 - z - x)^2 \, dx \, dz \\ &= \frac{1}{2} \int_0^1 \left[-\frac{(3 - z - x)^3}{3} \right]_{x=2z}^{x=3-z} dz = \frac{1}{6} \int_0^1 (3 - 3z)^3 \, dz \\ &= \frac{3^3}{6} \left[-\frac{(1 - z)^4}{4} \right]_0^1 = \frac{3^3}{24} = \frac{9}{8} \, . \end{aligned}$$

[Method 2] Consider the change of variables

$$u = 3z \, , \quad v = x + z \, , \quad w = x + y + z \, ,$$

which transforms E into a region in the uvw-space given by

$$\begin{aligned} E &\cong \left\{ (u, v, w) \in \mathbb{R}^3 \,\middle|\, 0 \le u \le v \le w \le 3 \right\} \\ &= \left\{ (u, v, w) \in \mathbb{R}^3 \,\middle|\, 0 \le w \le 3 \, , \, 0 \le v \le w \, , \, 0 \le u \le v \right\} \, . \end{aligned}$$

The required Jacobian can be calculated from

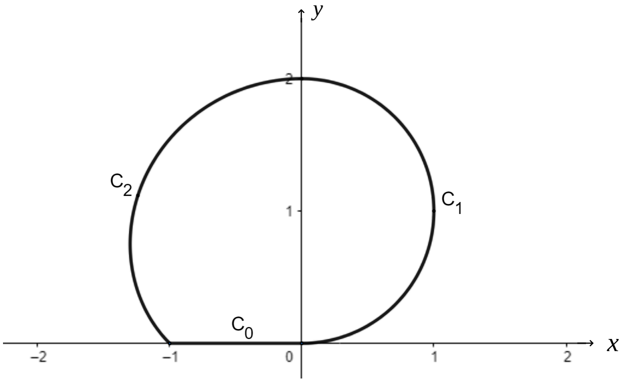
$$\frac{\partial(u, v, w)}{\partial(x, y, z)} = \begin{vmatrix} 0 & 0 & 3 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{vmatrix} = 3 \quad \Rightarrow \quad \frac{\partial(x, y, z)}{\partial(u, v, w)} = \frac{1}{3} \, .$$

Therefore, the total mass is found to be

$$
\begin{aligned}
M &= \iiint_E y \; dV = \int_0^3 \int_0^w \int_0^v (w-v)\,\frac{1}{3}\; du\; dv\; dw \\
&= \frac{1}{3}\int_0^3 \int_0^w \left(wv - v^2\right)\; dv\; dw \\
&= \frac{1}{3}\int_0^3 \left(\frac{w^3}{2} - \frac{w^3}{3}\right)\; dw = \frac{1}{3\cdot 6}\int_0^3 w^3\; dw \\
&= \frac{3^4}{3\cdot 6\cdot 4} = \frac{9}{8}\,.
\end{aligned}
$$

(calculation + answer: 3 points)

4. (12 points) Evaluate the line integral $\int_C \sin \pi x \; dx + (e^{y^2} + x^2) dy$ along the following choices of the curve C.



(a) (4 points) $C = C_0$ is the line segment from $(-1, 0)$ to $(0, 0)$.

(b) (8 points) $C = C_1 \cup C_2$, where C_1 is the polar curve $r = 2\sin\theta$, $0 \le \theta \le \frac{\pi}{2}$ and C_2 is the cardioid $r = 1 + \sin\theta$, $\frac{\pi}{2} \le \theta \le \pi$.

Solution:

(a) Describe C_0 as $r(t) = (t-1, 0)$, $0 \le t \le 1$ (寫出C_0的參數式得**1**分). Then

$$\begin{aligned}&\int_{C_0} \sin(\pi x)\; dx + (e^{y^2} + x^2)\; dy \\ &= \int_0^1 \sin(\pi(t-1)) \cdot 1 + (1 + (t-1)^2) \cdot 0 \; dt \text{ (用參數式代換線積分，得1分)} \\ &= \int_0^1 \sin(\pi(t-1))\; dt = \frac{-1}{\pi} \cos(\pi(t-1))\Big|_0^1 = \frac{-2}{\pi} \text{ (算出正確答案，得2分)}.\end{aligned}$$

(b) If D is the region bounded by C_0, C_1, and C_2, then by Green's theorem, we have

$$\int_{C_0 \cup C_1 \cup C_2} \sin(\pi x)\; dx + (e^{y^2} + x^2)\; dy = \iint_D 2x\; dA \text{ (使用 Green's theorem，得3分)}$$

$$= \int_0^{\pi/2} \int_0^{2\sin\theta} 2 \cdot r\cos\theta \cdot r \; dr d\theta + \int_{\pi/2}^{\pi} \int_0^{1+\sin\theta} 2 \cdot r\cos\theta \cdot r\; dr d\theta$$

(用參數式代換面積分，得**3**分，沒完全列正確的話，斟酌給分)

$$= \frac{4}{3}(\sin\theta)^4\Big|_0^{\pi/2} + \frac{1}{6}(1+\sin\theta)^4\Big|_{\pi/2}^{\pi} = \frac{4}{3} + \frac{-5}{2} = \frac{-7}{6} \text{ (成功算出面積分，得2分)}.$$

Therefore,

$$\int_{C_1 \cup C_2} \sin(\pi x)\; dx + (e^{y^2} + x^2)\; dy = \frac{-7}{6} + \frac{2}{\pi}.$$

5. (16 points) Let $\boldsymbol{F}(x,y) = P(x,y)\,\boldsymbol{i} + Q(x,y)\,\boldsymbol{j}$, where $P(x,y) = \dfrac{xy}{\sqrt{x^2+y^2}}$, $Q(x,y) = \dfrac{x^2+2y^2}{\sqrt{x^2+y^2}}$.

(a) (3 points) Compute $\dfrac{\partial P}{\partial y}$ and $\dfrac{\partial Q}{\partial x}$. Is $\boldsymbol{F}$ conservative on the right half plane $D = \{(x,y)|x > 0\}$? Justify your answer.

(b) (4 points) Compute $\int_C \boldsymbol{F} \cdot d\boldsymbol{r}$, where C is any curve in the right half plane D from $(1,1)$ to $(2,2)$.

(c) (4 points) Compute $\oint_C \boldsymbol{F} \cdot d\boldsymbol{r}$, where C is a positively oriented circle centered at $(0,0)$ with radius $r > 0$.

(d) (4 points) Compute $\oint_C \boldsymbol{F} \cdot d\boldsymbol{r}$, where C is any positively oriented simple closed curve, $C \subset \mathbb{R}^2 \backslash \{(0,0)\}$. (Hint: You need to discuss whether C encloses $(0,0)$ or not.)

(e) (1 point) Is $\boldsymbol{F}$ conservative on $\mathbb{R}^2\backslash\{(0,0)\}$? Justify your answer.

Solution:

(a)

$$\frac{\partial P}{\partial y} = \frac{x\sqrt{x^2+y^2} - \frac{1}{2}xy\frac{2y}{\sqrt{x^2+y^2}}}{x^2+y^2} = \frac{x\sqrt{x^2+y^2} - \frac{xy^2}{\sqrt{x^2+y^2}}}{x^2+y^2} = \frac{x^3+xy^2-xy^2}{(x^2+y^2)^{\frac{3}{2}}} = \frac{x^3}{(x^2+y^2)^{\frac{3}{2}}} \quad (1\%)$$

$$\frac{\partial Q}{\partial x} = \frac{2x\sqrt{x^2+y^2} - (x^2+2y^2)\frac{1}{2}\frac{2x}{\sqrt{x^2+y^2}}}{x^2+y^2} = \frac{2x^3+2xy^2-x^3-2xy^2}{(x^2+y^2)^{\frac{3}{2}}} = \frac{x^3}{(x^2+y^2)^{\frac{3}{2}}} \quad (1\%)$$

Since $\dfrac{\partial P}{\partial y} = \dfrac{\partial Q}{\partial x}$ on $D = \{(x,y)|x > 0\}$ and D is simply-connected, by Green's theorem $\vec{F}$ is conservative on D. (1%)

(b)

Let $\vec{r}(t) = \langle t, t\rangle,\ 1 \le t \le 2$. (1%)

$$\begin{aligned}\int_{(1,1)}^{(2,2)} \vec{F}\cdot d\vec{r} &= \int_1^2 \langle \frac{t^2}{\sqrt{2}t}, \frac{3t^2}{\sqrt{2}t}\rangle \cdot \langle 1,1\rangle dt \quad (2\%)\\ &= \sqrt{2}t^2|_1^2 = 3\sqrt{2} \quad (1\%)\end{aligned}$$

(c)

$\vec{r}(t) = \langle r\cos t, r\sin t\rangle,\ 0 \le t \le 2\pi$ (1%)

$$\begin{aligned}\oint_C \vec{F}\cdot d\vec{r} &= \int_0^{2\pi} \langle \frac{r^2\cos t\sin t}{r}, \frac{r^2(1+\sin^2 t)}{r}\rangle \cdot \langle -r\sin t, r\cos t\rangle dt \ (2\%)\\ &= \int_0^{2\pi} (-r^2\cos t\sin^2 t + r^2(1+\sin^2 t)\cos t)dt = \int_0^{2\pi} r^2\cos t dt = 0 \quad (1\%)\end{aligned}$$

(d)

Case 1 (2%): For any simple closed curve C in $\mathbb{R}^2\backslash\{(0,0)\}$ that enclosed $(0,0)$. Let D be the region bounded between C and C_r, and $\vec{F}$ is defined on D and $\dfrac{\partial P}{\partial y}, \dfrac{\partial Q}{\partial x}$ are continuous on D. By Green's theorem,

$$\iint_D (\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y})dA = 0 = \oint_C \vec{F}\cdot d\vec{r} - \oint_{C_r} \vec{F}\cdot d\vec{r}$$

for some r small enough such that C_r is inside C. Therefore,

$$\oint_C \vec{F}\cdot d\vec{r} = \oint_{C_r} \vec{F}\cdot d\vec{r} = 0 \ by \ (c)$$

where C_r is a circle with $x^2 + y^2 = r^2$.
Case 2 (2%): For any simple closed curve C in $\mathbb{R}^2\backslash\{(0,0)\}$ that does not enclosed $(0,0)$. Let D be the region bounded by C. By Green's theorem,

$$\iint_D (\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}) dA = \iint_D 0 dA = \oint_C \vec{F} \cdot d\vec{r} = 0$$

(e) (1%)
Yes, since $\oint_C \vec{F} \cdot d\vec{r} = 0$ for any simple close curve C in $\mathbb{R}^2\backslash\{(0,0)\}$, we get $\vec{F}$ is conservative in $\mathbb{R}^2\backslash\{(0,0)\}$, which is connected.

6. (10 points) Find the area of the part of the surface $x^2 + y^2 + z^2 = 1$ that lies within the cylinder $x^2 + y^2 + x = 0$ and above $z = 0$.

Solution:

Solution I. The equation of the cylinder is $\left(x + \frac{1}{2}\right)^2 + y^2 = \frac{1}{4}$. Set $D = \left\{(x, y) : \left(x + \frac{1}{2}\right)^2 + y^2 = \frac{1}{4}\right\}$.

$$\text{Area} = \iint_D \sqrt{1 + z_x^2 + z_y^2}\, dA$$

We compute $z_x = -\frac{x}{\sqrt{1 - x^2 - y^2}}$ and $z_y = -\frac{y}{\sqrt{1 - x^2 - y^2}}$ and use polar coordinates to obtain

$$\begin{aligned}\text{Area} &= \iint_D \frac{1}{\sqrt{1 - x^2 - y^2}}\, dA = \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \int_0^{-\cos\theta} \frac{1}{\sqrt{1 - r^2}}\, r\, dr\, d\theta = \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \left[-\sqrt{1 - r^2}\right]_0^{-\cos\theta} d\theta \\ &= \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} 1 - |\sin\theta|\, d\theta = 2 \cdot \int_{\frac{\pi}{2}}^{\pi} 1 - \sin\theta\, d\theta = \pi - 2.\end{aligned}$$

By symmetry, one may consider $D = \left\{(x, y) : \left(x - \frac{1}{2}\right)^2 + y^2 = \frac{1}{4}\right\}$ and compute likewise.

Solution II. The surface is parametrized by

$$r(u, v) = (\sin u \cos v, \sin u \sin v, \cos u), \quad 0 \le u \le \frac{\pi}{2}, \quad \frac{\pi}{2} + u \le v \le \frac{3\pi}{2} - u.$$

We compute $|r_u \times r_v| = \sin u$ and integrate by parts to obtain

$$\text{Area} = \int_0^{\frac{\pi}{2}} \int_{\frac{\pi}{2}+u}^{\frac{3\pi}{2}-u} \sin u\, dv\, du = \int_0^{\frac{\pi}{2}} (\pi - 2u) \sin u\, du = \left[-(\pi - 2u)\cos u - 2\sin u\right]_0^{\pi/2} = \pi - 2.$$

Grading scheme
3 points for the region, 5 points for the integrand, 1 point for calculation and 1 point for the correct answer.

(01-02班) Suppose that $f(x,y,z)$ is a scalar function with continuous second partial derivatives. Fix a point $P_0 = (x_0, y_0, z_0)$. Consider spheres S_ρ centered at P_0 with radius $\rho > 0$.

(a) (2 points) Parametrize S_ρ with spherical coordinates $\boldsymbol{r}(\varphi,\theta) = (x_0+\rho\sin\varphi\cos\theta, y_0+\rho\sin\varphi\sin\theta, z_0+\rho\cos\varphi)$, $0 \le \varphi \le \pi$, and $0 \le \theta \le 2\pi$. Write down the double integral in φ and θ that represents the average value of f on S_ρ.

(b) (4 points) Let function $A(\rho)$ be the average value of f on S_ρ, for $\rho > 0$. Evaluate $A'(\rho)$ in terms of $\iint_{S_\rho} \nabla f \cdot d\boldsymbol{S}$.

(c) (4 points) If $\nabla^2 f = f_{xx}+f_{yy}+f_{zz}$ is always positive, show that $A(\rho)$ is increasing. If $\nabla^2 f(x,y,z) = 0$ for all (x,y,z), compute $A(\rho)$.

Solution:

1. $\begin{cases} \mathbf{r}_\phi = (\rho\cos\phi\cos\theta, \rho\cos\phi\sin\theta, -\rho\sin\phi) \\ \mathbf{r}_\theta = (-\rho\sin\phi\sin\theta, \rho\sin\phi\cos\theta, 0) \end{cases}$

   $\Rightarrow \mathbf{r}_\phi \times \mathbf{r}_\theta = \rho^2\sin\phi(\sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi)$ (1 point)

   $A(\rho) = \dfrac{1}{\text{Area}(S_\rho)} \iint_{S_\rho} f(x,y,z)dS = \dfrac{1}{4\pi\rho^2}\int_0^{2\pi}\int_0^{\pi} f(\mathbf{r}(\phi,\theta))\,|\mathbf{r}_\rho \times \mathbf{r}_\theta|\,d\phi d\theta$

   $\because \mathbf{r}(\phi,\theta)$ is Spherical coordinate, $\therefore |\mathbf{r}_\phi \times \mathbf{r}_\theta| = \rho^2\sin\phi$

   $\Rightarrow A(\rho) = \dfrac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi} f(\mathbf{r}(\phi,\theta))\sin\phi\ d\phi d\theta$ (1 point)

2. $A'(\rho) = \dfrac{d}{d\rho}\left(\dfrac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi} f(\mathbf{r}(\phi,\theta))\sin\phi\ d\phi d\theta\right) = \dfrac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi}\dfrac{d}{d\rho}f(\mathbf{r}(\phi,\theta))\sin\phi\ d\phi d\theta$

   $= \dfrac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi}\left(\nabla f(\mathbf{r}(\phi,\theta)) \cdot \dfrac{d}{d\rho}\mathbf{r}(\phi,\theta)\right)\sin\phi\ d\phi d\theta$ (2 points)

   $\because \dfrac{d}{d\rho}\mathbf{r}(\phi,\theta) = (\sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi)$, (1 point)

   $\therefore A'(\rho) = \dfrac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi}\nabla f(\mathbf{r}(\phi,\theta)) \cdot \dfrac{\mathbf{r}_\phi \times \mathbf{r}_\theta}{\rho^2}\ d\phi d\theta = \dfrac{1}{4\pi\rho^2}\iint_{S_\rho}\nabla f \cdot d\mathbf{S}$ (1 point)

3. By Divergence Theorem,

   $A'(\rho) = \dfrac{1}{4\pi\rho^2}\iint_{S_\rho}\nabla f \cdot d\mathbf{S} = \dfrac{1}{4\pi\rho^2}\iiint_{B_\rho}\text{div}(\nabla f)dV$ (1 point)

   $= \iiint_{B_\rho}\nabla \cdot (\nabla f)\ dV = \iiint_{B_\rho}\nabla^2 f\ dV > 0$, where B_ρ is the ball centered at P_0 with radius ρ.

   That is, $B_\rho = \left\{(x,y,z) \in \mathbb{R}^3 : x^2+y^2+z^2 \le \rho\right\}$

   $\Rightarrow A(\rho)$ is increasing, if $\nabla^2 f$ is always positive. (1 point)

   If $\nabla^2 f = 0$ for all (x,y,z), then $A'(\rho) = \iiint_{B_\rho}\nabla^2 f\ dV = 0$,

   that is $A(\rho)$ is a constant. (1 point)

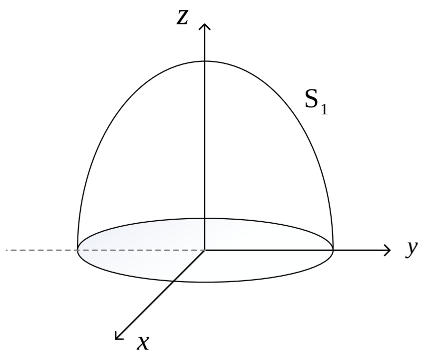
   $\Rightarrow A(\rho) = A(0) = \dfrac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi} f(x_0,y_0,z_0)\sin\phi\ d\phi d\theta = f_(x_0,y_0,z_0)$ (1 point)

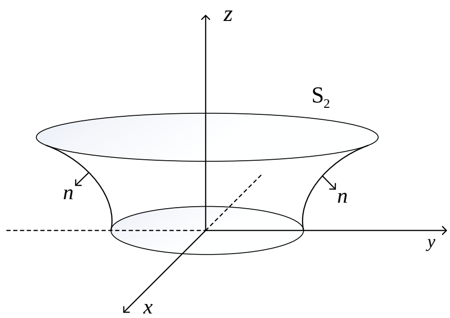
7. (14 points) Let $\boldsymbol{F} = (x - y)\boldsymbol{i} + (y - z)\boldsymbol{j} + (z - x)\boldsymbol{k}$ be a vector field on $\mathbb{R}^3$.

(a) (2 points) Compute $\operatorname{curl} \boldsymbol{F}$ on $\mathbb{R}^3$.

(b) (6 points) Let S_1 be a parametric surface given by $\boldsymbol{r}(r, \theta) = r\cos\theta\boldsymbol{i} + 2r\sin\theta\boldsymbol{j} + (9 - r^2)\boldsymbol{k}$ for $r \in [0, 3]$ and $\theta \in [0, 2\pi]$, which comes with the standard orientation given by the normal vector $\boldsymbol{r}_r \times \boldsymbol{r}_\theta$. Find the flux of $\operatorname{curl} \boldsymbol{F}$ across S_1.

(c) (6 points) Let S_2 be a surface defined by the equation $\dfrac{x^2}{9} + \dfrac{y^2}{36} - z^2 = 1$ for $z \in [0, 1]$ and endowed with the orientation given by the downward normal vector. Find the flux of $\operatorname{curl} \boldsymbol{F}$ across S_2.





Solution:

(a)

$$\operatorname{curl} \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x - y & y - z & z - x \end{vmatrix} = (1, 1, 1)$$

$$\operatorname{div} F = 3$$

(b) Method 1: Direct Calculation

$$\begin{aligned} \vec{r}(r, \theta) &= (r\cos\theta, 2r\sin\theta, 9 - r^2) \\ \vec{r_r} &= (\cos\theta, 2\sin\theta, -2r) \\ \vec{r_\theta} &= (-r\sin\theta, 2r\cos\theta, 0) \\ \vec{r_r} \times \vec{r_\theta} &= (4r^2\cos\theta, 2r^2\sin\theta, 2r) \end{aligned}$$

$$\begin{aligned} \therefore \iint_{S_1} \operatorname{curl} \vec{F} \cdot d\vec{S} &= \iint_{S_1} \operatorname{curl} \vec{F} \cdot (\vec{r_r} \times \vec{r_\theta}) dS \\ &= \int_0^{2\pi} \int_0^3 4r^2\cos\theta + 2r^2\sin\theta + 2r dr d\theta \\ &= 18\pi \end{aligned}$$

Method 2: Stokes' Theorem

At $z = 0, r = 3,\ \vec{r} = (3\cos\theta, 6\sin\theta, 0)$,

$C = \{(x, y, z) \big| x = 3\cos\theta, y = 6\sin\theta, z = 0, \text{ where } \theta \in [0, 2\pi]\}$

$$\begin{aligned} \therefore \iint_{S_1} \operatorname{curl} \vec{F} \cdot d\vec{S} &= \oint_C \vec{F} \cdot d\vec{r} \\ &= \int_0^{2\pi} 27\sin\theta\cos\theta + 18\sin^2\theta d\theta \\ &= 18\pi \end{aligned}$$

Method 3: Stokes' Theorem

Let $S_3 = \{(x,y,z) \Big| \frac{x^2}{9} + \frac{y^2}{36} \le 1, z = 0\}$ where $\vec{n} = \vec{k}$

$$\begin{aligned} \therefore \iint_{S_1} \operatorname{curl}\vec{F}\cdot d\vec{S} = \oint_C \vec{F}\cdot d\vec{r} &= \iint_{S_3} \operatorname{curl}\vec{F}\cdot d\vec{S} \\ &= \iint_{S_1} \operatorname{curl}\vec{F}\cdot \vec{k} dS \\ &= \iint_{S_3} 1 dS \\ &= 18\pi \text{ (Area of ellipse)} \end{aligned}$$

(c) **Solution 1.** (Direct Compute)

$S_2 := r(z,\theta) = \langle 3\sqrt{1+z^2}\cos\theta,\ 6\sqrt{1+z^2}\sin\theta,\ z\rangle,$

$0 \le z \le 1,\ 0 \le \theta \le 2\pi$ (1 point)

$r_z = \langle \frac{3z}{\sqrt{1+z^2}}\cos\theta,\ \frac{6z}{\sqrt{1+z^2}}\sin\theta,\ 1\rangle$

$r_\theta = \langle -3\sqrt{1+z^2}\sin\theta,\ 6\sqrt{1+z^2}\cos\theta,\ 0\rangle$ (1 point)

$r_z \times r_\theta = \langle -6\sqrt{1+z^2}\cos\theta,\ -3\sqrt{1+z^2}\sin\theta,\ 18z\rangle$ (1 point)

Take the negative-z direction, thus

$\iint_{S_2} \operatorname{curl}\mathbf{F}\cdot d\mathbf{S} = \iint_D \operatorname{curl}\mathbf{F}\cdot[-(r_z\times r_\theta)]dA$ (1 point)

$= \int_0^{2\pi}\int_0^1 6\sqrt{1+z^2}\cos\theta + 3\sqrt{1+z^2}\sin\theta - 18z\, dz\, d\theta$

$= \int_0^{2\pi}\int_0^1 -18z\, dz\, d\theta$

$= -18\pi.$ (2 points)

Solution 2. (Use Stokes' theorem once)

Let C_1 be the boundary of the oval $\frac{x^2}{9} + \frac{y^2}{36} = 2,\ z = 1$ and C_2 be the boundary of the oval $\frac{x^2}{9} + \frac{y^2}{36} = 1,\ z = 0.$

By Stokes' theorem, we know that

$\iint_{S_2} \operatorname{curl}\mathbf{F}\cdot\ d\mathbf{S} = \oint_{C_1} \mathbf{F}\cdot\ d\mathbf{r} + \oint_{C_2} \mathbf{F}\cdot\ d\mathbf{r}$

where C_1 is negative oriented, C_2 is positive oriented. (2 points)

Thus, we have

$r_1 = \langle 3\sqrt{2}\cos\theta,\ 6\sqrt{2}\sin\theta,\ 1\rangle, \qquad 0 \le \theta \le -2\pi$

$r_2 = \langle 3\cos\theta,\ 6\sin\theta,\ 0\rangle, \qquad 0 \le \theta \le 2\pi.$ (2 points)

$\iint_{S_2} \operatorname{curl}\mathbf{F}\cdot d\mathbf{S}$

$= \int_0^{-2\pi} \langle 3\sqrt{2}\cos\theta - 6\sqrt{2}\sin\theta,\ 6\sqrt{2}\sin\theta - 1,\ 1 - 3\sqrt{2}\cos\theta\rangle$

$\cdot\langle -3\sqrt{2}\sin\theta,\ 6\sqrt{2}\cos\theta,\ 0\rangle\, d\theta$

$+ \int_0^{2\pi} \langle 3\cos\theta - 6\sin\theta,\ 6\sin\theta,\ 3\cos\theta\rangle\cdot\langle -3\sin\theta,\ 6\cos\theta,\ 0\rangle\, d\theta$

$= -36\pi + 18\pi = -18\pi.$ (2 points)

Solution 3. (Use Stokes' theorem twice)

Let C_1 be the boundary of the oval $D_1 = \{\frac{x^2}{9} + \frac{y^2}{36} = 2,\ z = 1\}$ and C_2 be the boundary of the oval $D_2 = \{\frac{x^2}{9} + \frac{y^2}{36} = 1,\ z = 0\}$. Then (2 points)

$$\iint_{S_2} \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = \oint_{C_1} \mathbf{F} \cdot d\mathbf{r} + \oint_{C_2} \mathbf{F} \cdot d\mathbf{r} = \iint_{D_1} \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} + \iint_{D_2} \operatorname{curl} \mathbf{F} \cdot d\mathbf{S}$$

where D_1 is oriented downward, D_2 is oriented upward. (2 points)

$\iint_{D_1} \operatorname{curl} \mathbf{F} \cdot d\mathbf{S}$
$= \iint_{D_1} \langle 1,\ 1,\ 1\rangle \cdot \langle 0,\ 0,\ -1\rangle \, dA$
$= -\iint_{D_1} dA$
$= -A(D_1)$.

Where $A(D)$ is the area of the region D.

Similarly,

$\iint_{D_2} \operatorname{curl} \mathbf{F} \cdot d\mathbf{S}$
$= A(D_2)$.

$\iint_{S_2} \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = -A(D_1) + A(D_2)$
$= -\pi \cdot 3\sqrt{2} \cdot 6\sqrt{2} + \pi \cdot 3 \cdot 6$
$= -18\pi$ (2 points)

Note:

1. If you do wrong on orientation, the most score you get is 3.

(01-02班) Let $\mathbf{F} = (x-y)\mathbf{i} + (y-z)\mathbf{j} + (z-x)\mathbf{k}$ be a vector field on $\mathbb{R}^3$.

(b) (6 points) Let S_1 be the part of paraboloid $z = x^2 + (y-1)^2$ thea is below the plane $z = 5-2y$ with downward orientation. Find the flux of $\mathbf{F}$ across S_1, $\iint_{S_1} \mathbf{F}\cdot d\mathbf{S}$.

Solution:

(Method I) Let E is the solid bounded by the paraboloid $z = x^2+(y-1)^2$ and the plane $z = 5-2y$. Then by Divergence Theorem, the flux of $\mathbf{F}$ across the boundary of E, (2 points)

$$\iint_{S_1} \mathbf{F}\cdot d\mathbf{S} + \iint_{S'} \mathbf{F}\cdot d\mathbf{S} = \iiint_E \operatorname{div}\mathbf{F}\ dV \Rightarrow \iint_{S_1} \mathbf{F}\cdot d\mathbf{S} = \iiint_E 3dV - \iint_{S'} \mathbf{F}\cdot d\mathbf{S}$$

$\because z = x^2 + (y-1)^2 = 5-2y \Rightarrow x^2+y^2 = 4$,

$\therefore$ the projection of the intersection of the paraboloid and the plane to $xy-$ plane is a circle centered at $(0,0)$ with radius 2.

$$\iiint_E dV = \int_0^{2\pi}\int_0^2\int_{r^2-2r\sin\theta+1}^{5-2r\sin\theta} rdzdrd\theta \text{ (By Cylindrical coordinate)}$$

$$= \int_0^{2\pi}\int_0^2 (5-2r\sin\theta)-(r^2-2r\sin\theta+1)rdzdrd\theta = 2\pi\cdot\int_0^2 r(4-r^2)dr = 8\pi \text{ (2 points)}$$

$$\iint_{S'} \mathbf{F}\cdot d\mathbf{S} = \iint_{x^2+y^2\le 4} \mathbf{F}\cdot(0,2,1)dA = \iint_{x^2+y^2\le 4} 2y-(5-2y)-xdA$$

$$= \iint_{x^2+y^2\le 4} 4y-x-5dA = -5\cdot 2^2\pi = -20\pi$$

$$\Rightarrow \iint_{S_1} \mathbf{F}\cdot d\mathbf{S} = \iiint_E 3dV - \iint_{S'} \mathbf{F}\cdot d\mathbf{S} = 3\cdot 8\pi + 20\pi = 44\pi \text{ (2 points)}$$

(b) (Method II) $S_1 : \mathbf{r}(u,v) = (u,v,u^2+(v-1)^2) \Rightarrow \mathbf{r}_u\times\mathbf{r}_v = (-2u,-2(v-1),1)$

$$\iint_{S_1} \mathbf{F}\cdot d\mathbf{S} = \iint_{u^2+v^2\le 4} \mathbf{F}\cdot(-\mathbf{r}_u\times\mathbf{r}_v)\,dudv \text{ (2 points)}$$

$$= \iint_{u^2+v^2\le 4} 2u(u-v)+2(v-1)(v-u^2-(v-1)^2)-(u^2+(v-1)^2-u)dudv$$

$$= \iint_{u^2+v^2\le 4} 2u^2+2v^2+2u^2-2(v-1)^3-u^2-(v-1)^2dudv \text{ (By Symmetirc)}$$

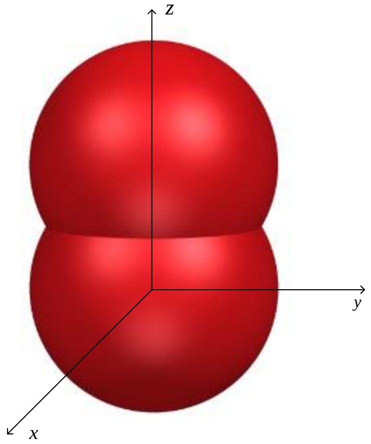
$$= \iint_{u^2+v^2\le 4} 3u^2+2v^2-2(v^3-3v^2+3v-1)-(v^2-2v+1)dudv$$

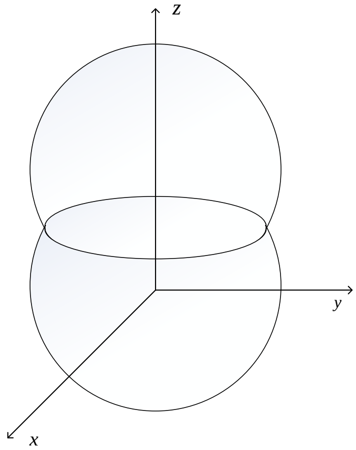
$$= \iint_{u^2+v^2\le 4} 3u^2+7v^2+1\ dudv = \int_0^{2\pi}\int_0^2 \left(3r^2\cos^2\theta+7r^2\sin^2\theta\right)r\ drd\theta + 4\pi \text{ (3 points)}$$

$$= \int_0^{2\pi}\int_0^2 5r^3\ drd\theta + 4\pi\left(\because \int_0^{2\pi}\sin^2\theta d\theta = \int_0^{2\pi}\cos^2\theta d\theta\right)$$

$$= 2\pi\cdot\frac{5}{4}r^4\Big|_0^2 + 4\pi = 44\pi \text{ (1 points)}$$

8. (12 points) Let S be the boundary surface of the union of the balls $x^2+y^2+z^2 \le 1$ and $x^2+y^2+(z-1)^2 \le 1$.





(a) (5 points) Use spherical coordinates to parametrize S.

(b) (7 points) Find $\iint_S \boldsymbol{F} \cdot d\boldsymbol{S}$ where $\boldsymbol{F} = \boldsymbol{i} + \boldsymbol{j} + z^2\,\boldsymbol{k}$ and S is given the outward orientation.

Solution:

(a) (sol 1.)

upper surface: $< 2\sin\phi\cos\phi\cos\theta, 2\sin\phi\cos\phi\sin\theta, 2\cos^2\phi >, 0 \le \theta \le 2\pi, 0 \le \phi \le \dfrac{\pi}{3}$

lower surface: $< \sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi >, 0 \le \theta \le 2\pi, \dfrac{\pi}{3} \le \phi \le \pi$

(sol 2.)

upper surface: $< 2\sin\phi\cos\phi\cos\theta, 2\sin\phi\cos\phi\sin\theta, \cos\phi + 1 >, 0 \le \theta \le 2\pi, 0 \le \phi \le \dfrac{2\pi}{3}$

lower surface: $< \sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi >, 0 \le \theta \le 2\pi, \dfrac{\pi}{3} \le \phi \le \pi$

score: the correct answer of upper surface gets 3 points, lower surface gets 2 points. Only write the correct Spherical coordinate, doesn't write the range of θ, ϕ gets 1 point separately. The correct range of θ on both surface gets 1 point. The correct range of ϕ on both surface gets 1 point separately.

(b) (sol 1.)

Use divergence theorem: $\int\int F dS = \int\int\int div F dV$

upper surface: $\displaystyle\int_0^{2\pi}\int_0^{\frac{\pi}{3}}\int_0^{2\cos\phi} 2\rho\cos\phi\rho^2\sin\phi d\rho d\phi d\theta = \frac{21\pi}{8}$

lower surface: $\displaystyle\int_0^{2\pi}\int_{\frac{\pi}{3}}^{\pi}\int_0^{1} 2\rho\cos\phi\rho^2\sin\phi d\rho d\phi d\theta = \frac{-3\pi}{8}$

total: $\dfrac{9\pi}{4}$

(sol 2.)

upper surface: $J = \begin{vmatrix} i & j & k \\ 2\cos(2\phi)\cos\theta & 2\cos(2\phi)\sin\theta & -2\sin(2\phi) \\ -\sin(2\phi)\sin\theta & \sin(2\phi)\cos\theta & 0 \end{vmatrix}$

$= < -2\sin^2(2\phi)\cos\theta, 2\sin^2(2\phi)\sin\theta, \sin(4\phi) >$

$\displaystyle\int_0^{2\pi}\int_0^{\frac{\pi}{3}} < 1, 1, 4\cos^4\phi >< -2\sin^2(2\phi)\cos\theta, 2\sin^2(2\phi)\sin\theta, \sin(4\phi) > d\phi d\theta$

$= \dfrac{87\pi}{32}$

lower surface: $J = \begin{vmatrix} i & j & k \\ \cos\phi\cos\theta & \cos\phi\sin\theta & -\sin\phi \\ -\sin\phi\sin\theta & \sin\phi\cos\theta & 0 \end{vmatrix}$

$= < \sin^2\phi\cos\theta, \sin^2\phi\sin\theta, \sin\phi\cos\theta >$

$\int_0^{2\pi}\int_{\frac{\pi}{3}}^{\pi} < 1, 1, \cos^2\phi >< \sin^2\phi\cos\theta, \sin^2\phi\sin\theta, \sin\phi\cos\theta > d\phi d\theta$

$= \dfrac{-15\pi}{32}$

total: $\dfrac{9\pi}{4}$

score: Knowing to use divergence theorm gets 2 points, other method 1point. Use the upper answer of Spherical coordinate($\rho^2\sin\phi$) gets 2 points. The correct interval of integral gets 1 point separately, however you do write the correct value of integral.