微乙小考五 (2017/12/28)

1. (7分)

(1) 設 $y=f(x)$ 在 $[a,b]$ 滿足 $f(x)$ 可微且 $f'(x)$ 連續, 寫下 $y=f(x)$ 從 $x=a$ 到 $x=b$ 之弧長公式.(3%)

(2) 若 $f(x)=\dfrac{e^x+e^{-x}}{2}$, 試求 $y=f(x)$ 從 $x=0$ 到 $x=1$ 之弧長.(4%)

sol: (1)

$$L=lim\ \sum\sqrt{(\Delta x)^2+(\Delta y)^2}=lim\ \sum\sqrt{1+(\frac{\Delta y}{\Delta x})^2}\Delta x=\int_a^b\sqrt{1+(f'(x))^2}dx$$

(2)

將 $f(x)=\dfrac{e^x+e^{-x}}{2}$ 帶入公式 (1) 中，考慮 [0,1]

$$f'(x)=\frac{e^x-e^{-x}}{2}\Rightarrow\sqrt{1+(f'(x))^2}=\sqrt{1+\frac{e^{2x}+e^{-2x}-2}{4}}=\sqrt{\frac{e^{2x}+e^{-2x}+2}{4}}=\frac{e^x+e^{-x}}{2}$$

so we have

$$L=\int_a^b\sqrt{1+(f'(x))^2}dx=\int_0^1\frac{e^x+e^{-x}}{2}dx=(\frac{e^x-e^{-x}}{2})_{x=0}^{x=1}=\frac{e-e^{-1}}{2}$$

2. (6分) 當 $-1<x<1$, 試求 $\dfrac{x}{(1-x)^2}$ 之無窮級數展開.(提示: 考慮 $(\dfrac{1}{1-x})'$ 並且乘上 x.)

sol:

$$\int\frac{1}{(1-x)^2}dx=c+\frac{1}{1-x}=c+1+x+x^2+x^3+...=c+\sum_{n=0}^{\infty}x^n$$

Since $-1<x<1$ the series is uniformly converge, and apply Taylor's inequality theorem. Because the series is uniformly converge we can derivative term by term. So we have

$$\frac{1}{(1-x)^2}=\frac{d}{dx}(c+\sum_{n=0}^{\infty}x^n)=\frac{d}{dx}(c+1+x+x^2+x^3+...)=1+2x+3x^2+...=\sum_{n=0}^{\infty}(n+1)x^n$$

Again, the series is uniformly converge we can multiply x both side. So we have

$$\frac{x}{(1-x)^2}=x\sum_{n=0}^{\infty}(n+1)x^n=\sum_{n=0}^{\infty}(n+1)x^{n+1}$$

3. (7分)

(1) 寫下 $f(x)$ 在 $x=a$ 之2階泰勒多項式 $P_2(x)$. (3%)

(2) 在(1)中, 設 $f(x)=e^{x^2}$ 與 $a=1$. 試求 $P_2(x)$.(4%)

sol: (1)

$$P_2(x)=f(a)+\frac{f'(a)}{1!}(x-a)+\frac{f''(a)}{2!}(x-a)^2$$

(2)

Put $f(x)=e^{x^2}$, $a=1$ in (1) , $f'(x)=2xe^{x^2}$ and $f''(x)=2e^{x^2}+4x^2e^{x^2}$

$$P_2(x)=f(1)+\frac{f'(1)}{1!}(x-1)+\frac{f''(1)}{2!}(x-1)^2=e+2e(x-1)+\frac{2e+4e}{2!}(x-1)^2=3ex^2-4ex+2e$$