微乙小考二 (2017/10/12)

1. 探討 $\lim\limits_{x\to\infty} \sin \pi x$. 之極限值。

   (1) (2 %) 計算 $\lim\limits_{n\to\infty} \sin 2n\pi$.

   (2) (2 %) 計算 $\lim\limits_{n\to\infty} \sin(2n+\frac{1}{2})\pi$.

   (3) (1%) 請問 $\lim\limits_{x\to\infty} \sin \pi x$ 是否存在?

sol: (1) $\sin 2n\pi$ is zero for any $n \in \mathbb{Z}$ such that $\lim\limits_{n\to\infty} \sin 2n\pi$ goes to 0.

(2) Similarly, $\forall n \in \mathbb{Z}$, $\sin(2n+\frac{1}{2})\pi = \sin(2n\pi + \frac{\pi}{2}) = \sin(\frac{\pi}{2}) = 1$. Thus, $\lim\limits_{n\to\infty} \sin(2n+\frac{1}{2})\pi = 1$.

(3) With discussion of (1), (2) and the periodicity of sine function, we cannot ultimately derive a convergent value by $\sin \pi x$ as its variable x is going to infinity. Therefore, $\lim\limits_{x\to\infty} \sin \pi x$ does not exist.

2. (9%) 試求下列之極限值。

   (1) $\lim\limits_{x\to 0} (\frac{1}{x\sqrt{1+x}} - \frac{1}{x})$. (2) $\lim\limits_{h\to 0} \frac{\frac{1}{x+h}-\frac{1}{x}}{h}$. (3) $\lim\limits_{t\to 1} \frac{t^4-1}{t^3-1}$.

sol: (1)
$$\lim_{x\to 0}\left(\frac{1}{x\sqrt{1+x}} - \frac{1}{x}\right) = \lim_{x\to 0}\left(\frac{1-\sqrt{1+x}}{x\sqrt{1+x}}\right) = \lim_{x\to 0}\left(\frac{1-\sqrt{1+x}}{x\sqrt{1+x}} \cdot \frac{1+\sqrt{1+x}}{1+\sqrt{1+x}}\right)$$
$$= \lim_{x\to 0} \frac{1-(1+x)}{(x\sqrt{1+x})(1+\sqrt{1+x})} = \lim_{x\to 0} \frac{-1}{(\sqrt{1+x})(1+\sqrt{1+x})} = \frac{-1}{(\sqrt{1+0})(1+\sqrt{1+0})} = \frac{-1}{2}.$$

(2)
$$\lim_{h\to 0} \frac{\frac{1}{x+h}-\frac{1}{x}}{h} = \lim_{h\to 0} \frac{x-(x+h)}{x(x+h)} \cdot \frac{1}{h} = \lim_{h\to 0} \frac{-h}{x(x+h)} \cdot \frac{1}{h} = \lim_{h\to 0} \frac{-1}{x(x+h)} = \frac{-1}{x^2}.$$

(3)
$$\lim_{t\to 1} \frac{t^4-1}{t^3-1} = \lim_{t\to 1} \frac{(t^2+1)(t+1)(t-1)}{(t-1)(t^2+t+1)} = \lim_{t\to 1} \frac{(t^2+1)(t+1)}{t^2+t+1} = \frac{(1+1)(1+1)}{1+1+1} = \frac{4}{3}.$$

3. (6%) 試求下列之極限值。

   (1) $\lim\limits_{x\to\infty} (1+\frac{4}{x})^x$.

   (2) $\lim\limits_{x\to\infty} (\frac{x+\frac{1}{2}}{x-\frac{1}{2}})^x$.

sol:
$$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = e$$

(1)
$$\lim_{x\to\infty}(1+\frac{4}{x})^x = \lim_{x\to\infty}\left[\left(1+\frac{4}{x}\right)^{\frac{x}{4}}\right]^4 = \left[\lim_{x'\to\infty}\left(1+\frac{1}{x'}\right)^{x'}\right]^4 = e^4.\ (x' \equiv x/4)$$

(2) Let $h \equiv x - \frac{1}{2}$, $(x\to\infty,\ h\to\infty)$

$$\lim_{x\to\infty}\left(\frac{x+1/2}{x-1/2}\right)^x = \lim_{h\to\infty}\left(\frac{h+1}{h}\right)^{h+\frac{1}{2}} = \lim_{h\to\infty}\left(1+\frac{1}{h}\right)^h \cdot \lim_{h\to\infty}\left(1+\frac{1}{h}\right)^{\frac{1}{2}} = e\cdot\sqrt{1} = e.$$