1061微乙01-05班期中考解答和評分標準

1. (10%)

   (a) 求 $\lim_{x\to 1}\frac{(x+1)(3^{x^2+1}-9)}{x^2-1}$.

   (b) 求 $\lim_{x\to 2}\frac{\tan^3(\pi(x-2))}{(x-2)^3 x}$.

Solution:

(a) (5 points) Method without L'Hospital rule:

$$\begin{aligned}\lim_{x\to 1}\frac{(x+1)(3^{x^2+1}-9)}{x^2-1} &= \lim_{x\to 1}(x+1)\lim_{x\to 1}\frac{3^{x^2+1}-3^2}{(x^2+1)-2} \text{ (for the latter let } y=x^2+1)\\ &= \lim_{x\to 1}(x+1)\lim_{y\to 2}\frac{3^y-3^2}{y-2}\\ &= 2\cdot\left.\frac{d(3^y)}{dy}\right|_{y=2}\\ &= 2\cdot 3^2\cdot\ln 3\\ &= 18\ln 3. \text{ (5 points)}\end{aligned}$$

Method with L'Hospital rule:

$$\begin{aligned}\lim_{x\to 1}\frac{(x+1)(3^{x^2+1}-9)}{x^2-1} &= \lim_{x\to 1}\frac{3^{x^2+1}-9}{x-1}\\ &= \lim_{x\to 1}\frac{3^{x^2+1}\cdot\ln 3\cdot 2x}{1}\\ &= 18\ln 3. \text{ (5 points)}\end{aligned}$$

註: When doing the derivative of $(3^{x^2+1})' = 3^{x^2+1}\cdot\ln 3\cdot 2x$ by chain rule, missing any one term of $\{3^{x^2+1}, \ln 3, 2x\}$ will cost you 2 points and missing any two terms of $\{3^{x^2+1}, \ln 3, 2x\}$ will cost you 5 points.

(b) (5 points)

$$\begin{aligned}\lim_{x\to 2}\frac{\tan^3(\pi(x-2))}{(x-2)^3 x} &= \lim_{x\to 2}\frac{\sin^3(\pi(x-2))}{(\pi(x-2))^3}\cdot\frac{\pi^3}{x\cos^3(\pi(x-2))}\\ &= \lim_{x\to 2}\frac{\sin^3(\pi(x-2))}{(\pi(x-2))^3}\lim_{x\to 2}\frac{\pi^3}{x\cos^3(\pi(x-2))} \text{ (for the former let } y=\pi(x-2))\\ &= \lim_{y\to 0}\frac{\sin^3 y}{y^3}\lim_{x\to 2}\frac{\pi^3}{x\cos^3(\pi(x-2))}\\ &= 1\cdot\frac{\pi^3}{2}\\ &= \frac{\pi^3}{2}. \text{ (5 points)}\end{aligned}$$

註: You will get no points if you write

$$\lim_{x\to 2}\frac{\tan(\pi(x-2))}{x-2} = 1 \text{ or } \lim_{x\to 2}\frac{\sin(\pi(x-2))}{x-2} = 1.$$

2. (10%)

(a) 陳述平均值定理。

(b) 證明 $\tan^{-1} y - \tan^{-1} x \leq y - x$ 對 $y \geq x \geq 0$ 都成立。

Solution:

(a) (5 points) Assume $f(x)$ is defined in $[a, b]$ and satisfies

(i) $f(x)$ is continuous in $[a, b]$;

(ii) $f(x)$ is differentiable in (a, b).

Then there exists $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}. \text{ (5 points)}$$

註1: You will get no points if you write

$$f(c) = \frac{f(b) - f(a)}{b - a}.$$

註2: You will get 3 points if you state Rolle's Theorem instead of Mean Value Theorem. □

(b) (5 points) We may assume $y > x$ since the case $y = x$ is obvious. Define $f(t) = \tan^{-1} t$ in $[x, y]$. Then $f(t)$ satisfies the assumptions of Mean Value Theorem. Hence there exists $c \in (x, y)$ such that

$$f'(c) = \frac{f(y) - f(x)}{y - x}.$$

That is,

$$\frac{1}{1 + c^2} = \frac{\tan^{-1} y - \tan^{-1} x}{y - x}. \text{ (3 points)}$$

Since $\frac{1}{1 + c^2} \leq 1$ and $y - x > 0$, we have

$$\tan^{-1} y - \tan^{-1} x \leq \frac{1}{1 + c^2}(y - x) \leq y - x. \text{ (2 points)}$$

註: For the last 2 points, one should specify $\frac{1}{1 + c^2} \leq 1$ instead of $\frac{1}{1 + c^2} \geq 0$ to obtain the final inequality.

3. (10%)

(a) 令 $h(x) = x^x$, 求 $h'(2)$.

(b) 令 $f(x) = 2^x$, 求 $f^{(5)}(x)$.

Solution:

(a)

$$\begin{aligned} h(x) &= e^{x \ln x} && (1\%) \\ h'(x) &= x^x \cdot (x \ln x)' && (1\%) \\ &= x^x \cdot (\ln x + 1) && (2\%) \\ h'(2) &= 4(\ln 2 + 1) && (1\%) \end{aligned}$$

(b)

$$\begin{aligned} f(x) &= e^{x \ln 2} && (1\%) \\ f'(x) &= 2^x \cdot \ln 2 && (1\%) \\ f''(x) &= \ln 2 \cdot (2^x)' \\ &= (\ln 2)^2 \cdot 2^x \\ &\vdots \\ f^{(5)}(x) &= (\ln 2)^5 \cdot 2^x && (3\%) \end{aligned}$$

4. (10%) 求 $\frac{d}{dx}(\cos \tan^{-1} x)$.

Solution:

Method 1

Let $f(x) = \cos x$ and $g(x) = \tan^{-1} x$,

$$\begin{aligned} \frac{d}{dx}(\cos \tan^{-1} x) &= \frac{d}{dx} f(g(x)) \\ &= f'(g(x)) \cdot \frac{d}{dx} g(x) && (2\%) \\ &= -\sin \tan^{-1} x \cdot \frac{d}{dx}(\tan^{-1} x) && (4\%) \\ &= -\sin \tan^{-1} x \cdot \frac{1}{x^2 + 1} && (4\%) \end{aligned}$$

Method 2

Let $f(x) = \cos \tan^{-1} x$ and $\theta = \tan^{-1} x$, where $\theta \in (-\pi/2, \pi/2)$. (2%)

For both cases of $\theta \in [0, \pi/2)$ and $\theta \in (-\pi/2, 0]$, we have:

$$\begin{aligned} f(x) &= \cos\theta = \frac{1}{\sqrt{x^2 + 1}} = (x^2 + 1)^{-1/2} && (2\%) \\ f'(x) &= \left(-\frac{1}{2}\right)(x^2 + 1)^{-3/2} \cdot 2x && (3\%) \\ &= \frac{-x}{(x^2 + 1)^{3/2}} && (3\%) \end{aligned}$$

5. (10%) 用線性逼近估計 $\sqrt{1+\sin(0.003)}$.

Solution:

設 $f(x) = \sqrt{1+\sin(x)}$ (2 pt)

$\Rightarrow f'(x) = \dfrac{\cos(x)}{2\sqrt{1+\sin(x)}}$ (3 pt)

$f(x) \approx L(x) = f(a) + f'(a)(x-a)$ (2 pt)

choose $a = 0$ (1 pt)

$\therefore f(0.003) \approx L(0.003) = f(0) + f'(0)(0.003-0) = 1 + \dfrac{0.003}{2} = 1.0015$

6. (14%) 用隱函數微分求 $\dfrac{dy}{dx}$ 及 $\dfrac{d^2y}{dx^2}$ 在曲線 $y^3 + xy + x^2y - x^3 = -1$ 的點 $x = 1, y = 0$ 上之值。

Solution:

等號兩邊對 x 微分得到

$3y^2\dfrac{dy}{dx} + y + x\dfrac{dy}{dx} + 2xy + x^2\dfrac{dy}{dx} - 3x^2 = 0$ —(式1) (5 pt)

將 $x = 1$, $y = 0$ 帶入上式得 $\dfrac{dy}{dx} = \dfrac{3}{2}$ (2 pt)

再將，式(1) 兩邊對 x 微分得到

$6y(\dfrac{dy}{dx})^2 + 3y^2\dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} + \dfrac{dy}{dx} + x\dfrac{d^2y}{dx^2} + 2y + 2x\dfrac{dy}{dx} + 2x\dfrac{dy}{dx} + x^2\dfrac{d^2y}{dx^2} - 6x = 0$ (5 pt)

將 $x = 1$, $y = 0$, $\dfrac{dy}{dx} = \dfrac{3}{2}$ 帶入上式得 $\dfrac{d^2y}{dx^2} = -\dfrac{3}{2}$ (2 pt)

7. (23%) 若 $y = f(x) = \dfrac{9(x^2-3)}{x^3}$.

(a) $y = f(x)$ 在 ____________________ (區間)遞增 (3%)

$y = f(x)$ 在 ____________________ (區間)遞減 (3%)

(b) $y = f(x)$ 在 ____________________ (區間)凹向上 (3%)

$y = f(x)$ 在 ____________________ (區間)凹向下 (3%)

(c) $y = f(x)$ 之極大值(若存在的話): ____________________ (座標) (2%)

$y = f(x)$ 之極小值(若存在的話): ____________________ (座標) (2%)

(d) $y = f(x)$ 之所有漸近線為 ____________________ (4%)

(e) 畫出 $y = f(x)$ 之圖形 (3%)

Solution:

- $y = f(x) = \dfrac{9(x^2-3)}{x^3}$

(a) (6 points) Increment: $(-3,0)\bigcup(0,3)$.

. Decrement: $(-\infty,-3)\bigcup(3,\infty)$.

$$f'(x) = \frac{-9x^2+81}{x^4}$$

$$f'(x) = 0 \Rightarrow x^4 \neq 0 \Rightarrow x \neq 0$$

. $$-9(x+3)(x-3) = 0 \Rightarrow x = 3 \text{ or } -3$$

Increment: $f'(x) < 0 \Rightarrow -3 < x < 3 \Rightarrow (-3,0)\bigcup(0,3)$

Decrement: $f'(x) < 0 \Rightarrow x > 3 \text{ or } x < -3 \Rightarrow (-\infty,-3)\bigcup(3,\infty)$

(b) (6points) concave upward: $(-3\sqrt{2},0)\bigcup(3\sqrt{2},\infty)$.

. concave downward: $(-\infty,-3\sqrt{2})\bigcup(0,3\sqrt{2})$.

$$f''(x) = \frac{18x^2-324}{x^5}$$

$$f''(x) = 0 \Rightarrow x^5 \neq 0 \Rightarrow x \neq 0$$

. $$18(x^2-18) = 0 \Rightarrow x = 3\sqrt{2} \text{ or } -3\sqrt{2}$$

concave upward: $f''(x) > 0 \Rightarrow x > 3\sqrt{2} \text{ or } x < -3\sqrt{2} \Rightarrow (-3\sqrt{2},0)\bigcup(3\sqrt{2},\infty)$.

concave downward: $f''(x) > 0 \Rightarrow -3\sqrt{2} < x < 3\sqrt{2} \Rightarrow (-\infty,-3\sqrt{2})\bigcup(0,3\sqrt{2})$.

- (a,b)score: answer 1 point separately(a didn't wirte $x \neq 0$ lose 0.5 point), right differential equations 1 point, how you find interval 1 points.

(c) (4points) local max: $\underline{(3, 2)}$, local mini:$\underline{(-3, -2)}$

local max: value of f of $f(c) \geq f(x)$ when x is near c.

local mini: value of f of $f(c) \leq f(x)$ when x is near c.

- score: answer 1 point separately, right concept 2 points.

(d) (4 points) asymptote:$\underline{x = 0, y = 0}$

Horizontal Asymptotes: $y = 0 \lim_{x\to\infty} \frac{9(x^2-3)}{x^3} = \lim_{x\to\infty} \frac{9}{x} + \lim_{x\to\infty} \frac{9(-3)}{x^3} \simeq 0 + 0 \Rightarrow 0$

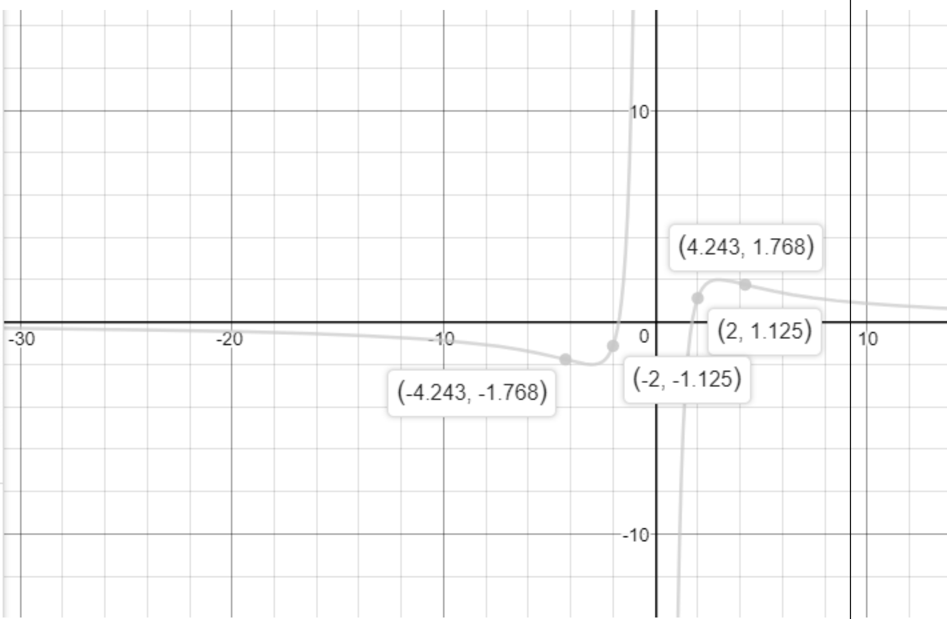
Vertical Asymptotes: $x = 0 \lim_{x\to 0} \frac{9(x^2-3)}{x^3} \simeq \lim_{x\to\infty} \frac{9(-3)}{x^3} \Rightarrow \infty$

Slant Asymptotes: $y = mx + k \Rightarrow \lim_{x\to\infty} \frac{f(x)}{x} = m \Rightarrow \lim_{x\to\infty} \frac{9(x^2-3)}{x^4} \simeq \lim_{x\to\infty} \frac{9(-3)}{x^4} \Rightarrow \infty$ (it's mean $y = \infty \Rightarrow x = 0$)

- score: answer 1 points separately, right concept 2 points. (notify: if you jump through process lose 1point.)

(f) (3 points) Plot the f(x) and indicate asymptote, local point and infection point.

x	$\frac{9(x^2-3)}{x^3}$
-2	-1.125
-1	18
0	undefined
1	-18
2	1.125
$-3\sqrt{(2)}$	-1.767767
$3\sqrt{(2)}$	1.767767



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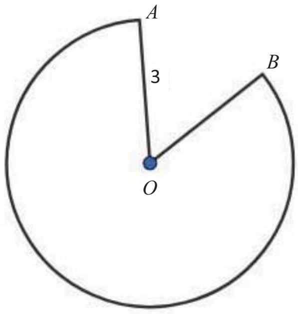
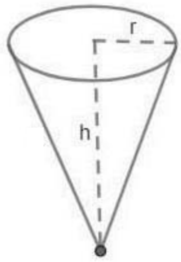
- vertical asymptote 0.5point, horiaontal asymptote 0.5point.
- infection points 0.5point separately, local points 0.5point spearately.

8. (13%) 半徑 3 之圓形紙，由圓心 O 截去一角 $\angle AOB$. 黏住 $\overline{OA}, \overline{OB}$ 作成錐形杯，求最大容量。錐之體積是 $\frac{\pi r^2 h}{3}$，其中 r =錐底半徑，h =錐高。





Solution:

已知錐體體積公式 $V = \frac{\pi r^2 h}{3}$且$r^2 + h^2 = 9, 0 \le h \le 3, 0 \le r \le 3$

1. (方法一) 將 $r^2 = 9 - h^2$ 帶入錐體體積公式得

$$V(h) = \frac{\pi(9-h^2)h}{3}$$

兩邊對 h 微分得

$$V'(h) = \frac{\pi(9-3h^2)}{3}$$

因為局部最大值可能發生在微分等於0的點，欲使 $V'(h) = 0$，h 必須滿足 $9 - 3h^2 = 0$，因此$h = \pm\sqrt{3}$，但$h = -\sqrt{3}$不合(因為$0 \le h \le 3$)。

使用二階檢測來檢查$h = \sqrt{3}$時是不是局部最大值(local maximum)，將$V'(h) = \frac{\pi(9-3h^2)}{3}$兩邊在對$h$微分得

$$V''(h) = -2\pi h$$

所以$V''(\sqrt{3}) < 0$，因此$V(\sqrt{3})$是一個局部最大值。

最後檢查邊界點，$V(0) = 0$且$V(3) = 0$皆小於$V(\sqrt{3})$，因此$V(\sqrt{3}) = 2\sqrt{3}\pi$為錐杯的最大體積。

2. (方法二) 將$h = \sqrt{9-r^2}$帶入錐體體積公式得

$$V(r) = \frac{\pi r^2\sqrt{9-r^2}}{3}$$

兩邊對r微分得

$$V'(r) = \frac{\pi}{3}(2r\sqrt{9-r^2} - \frac{r^3}{\sqrt{9-r^2}}) = \frac{\pi}{3}(18r - 3r^3)$$

因為局部最大值可能發生在微分等於0的點，欲使$V'(r) = 0$，r必須滿足$r(18-3r^2) = 0$，因此$r = \pm\sqrt{6}$或0，但$r = -\sqrt{6}$不合(因為$0 \le r \le 3$)。

使用一階檢測來檢查$r = 0$和$\sqrt{6}$是否為局部最大值，由$V'(r)$知當$-\sqrt{6} \le r \le \sqrt{6}$時$V'(r) \le 0$，且當$r \ge \sqrt{6}$時$V'(r) \ge 0$。因此$r = \sqrt{6}$為局部最大值。

最後檢查邊界點，$V(0) = 0$且$V(3) = 0$皆小於$V(\sqrt{6})$，因此$V(\sqrt{6}) = 2\sqrt{3}\pi$為錐杯的最大體積。

配分方式：(本題作法還有很多種，不管哪種都是以以下原則配分)

1. 正確寫出$V(h)$或$V(r)$得4分。
2. 正確地把上面的函數做一次微分得3分。
3. 解出一次微分等於0的點且說明負不合得2分，未說明負不合得1分。
4. 答案部份包含4分，斟酌扣分但最多扣到4分
   (a) 未說明為什麼微分等於0的點是局部最大值扣3分
   (b) 未檢查邊界點扣1分
   (c) 答案錯誤扣1分