1. (15%)
   (a) (5%) 請精確陳述微積分基本定理。
   (b) (10%) 函數 $f(x)$ 滿足
$$\int_0^x f(t)dt = \int_x^1 t^2 f(t)dt + \frac{x^{16}}{8} + \frac{x^{18}}{9} + C \qquad \text{對所有的 } x,$$
   其中 C 是常數。求出函數 $f(x)$ 及常數 C.

Solution:

(a) (5 points) Assume that $f(x)$ is continuous on $[a,b]$ (1 point). Then the following two statements hold true:

(Part I) Let $F(x) = \int_a^x f(t)dt$. Then $F'(x) = f(x)$. (2 points)

(Part II) If $G'(x) = f(x)$. Then $\int_a^b f(t)dt = G(b) - G(a)$. (2 points)

註: The fact that
$$F(x) = \int f(x)dx \text{ implies } F'(x) = f(x)$$
is just a definition for anti-derivatives but not a part of Fundamental Theorems of Calculus. You will get no points if you write down this as part I in your answer. □

(b) (10 points) Do the derivatives on the both sides of
$$\int_0^x f(t)dt = \int_x^1 t^2 f(t)dt + \frac{x^{16}}{8} + \frac{x^{18}}{9} + C,$$
we obtain
$$f(x) = -x^2 f(x) + 2x^{15} + 2x^{17}.$$
Hence
$$f(x) = \frac{2x^{15} + 2x^{17}}{1 + x^2} = 2x^{15}. \text{ (7 points)}$$
Furthermore, set $x = 0$ in the formula $\int_0^x f(t)dt = \int_x^1 t^2 f(t)dt + \frac{x^{16}}{8} + \frac{x^{18}}{9} + C$, we have
$$\begin{aligned} 0 &= \int_0^1 t^2 f(t)dt + C \\ &= \int_0^1 2t^{17}dt + C \\ &= \frac{t^{18}}{9}\Big|_0^1 + C \\ &= \frac{-1}{9} + C. \end{aligned}$$
Hence $C = \frac{-1}{9}$. (3 points)

註: Doing the derivatives incorrectly such as writing down
$$f(x) - f(0) = f(1) - x^2 f(x) + 2x^{15} + 2x^{17},$$
or making any similar mistakes in this step, you will get no more then 5 points for this question. □

2. (7%) 求 $\lim_{x\to\infty} \frac{\ln(1+x)}{\ln(1+x^2)}$.

Solution:

(7 points) By using L'Hospital rule (for the $\frac{\infty}{\infty}$ form), we have

$$\begin{aligned}\lim_{x\to\infty} \frac{\ln(1+x)}{\ln(1+x^2)} &= \lim_{x\to\infty} \frac{\frac{1}{1+x}}{\frac{2x}{1+x^2}} \\ &= \lim_{x\to\infty} \frac{x^2+1}{2x(x+1)} \quad \text{(5 points)} \\ &= \lim_{x\to\infty} \frac{x^2+1}{2x^2+2x} \\ &= \frac{1}{2}. \quad \text{(2 points)}\end{aligned}$$

註: Doing the derivatives as

$$\left(\ln(1+x^2)\right)' = \frac{1}{1+x^2}$$

you will get no points for this question. □

3. (7%) 求 $\lim_{x\to 0}\left(\frac{1}{x^2}-\frac{1}{\sin^2 x}\right)$.

Solution:

Solution 1.

$cosx = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + ...$

$cos2x = 1 - \frac{2^2 * x^2}{2!} + \frac{2^4 * x^4}{4!} - \frac{2^6 * x^6}{6!} + ...$

$sin^2x = \frac{1 - cos2x}{2} = \frac{2 * x^2}{2!} - \frac{2^3 * x^4}{4!} + \frac{2^5 * x^6}{6!} + ...$

$\lim_{x\to 0} \frac{\frac{2*x^2}{2!} - \frac{2^3*x^4}{4!} + \frac{2^5*x^6}{6!} + ... - x^2}{x^2 * (\frac{2*x^2}{2!} - \frac{2^3*x^4}{4!} + \frac{2^5*x^6}{6!} + ...)}$

$\lim_{x\to 0} \frac{-\frac{2^3*x^4}{4!} + \frac{2^5*x^6}{6!} + ...}{\frac{2*x^4}{2!} - \frac{2^3*x^6}{4!} + \frac{2^5*x^8}{6!} + ...} \Rightarrow \frac{\frac{-2}{3}}{2} = \frac{-1}{3}$

Solution 2.

$\lim_{x\to 0} \frac{sin^2x - x^2}{x^2 sin^2x}$ because equ. satisfy L'H $(\frac{0}{0})$

$\Rightarrow \lim_{x\to 0} \frac{sin2x - 2x}{2xsin^2x + x^2 sin2x}$ using equ. $sin2x = 2sinxcosx$ and L'H $(\frac{0}{0})$

$\Rightarrow \lim_{x\to 0} \frac{2cos2x - 2}{2sin^2x + 4xsin2x + 2x^2cos2x}$ using equ. L'H $(\frac{0}{0})$

$\Rightarrow \lim_{x\to 0} \frac{-4sin2x}{6sin2x + 12xcos2x - 4x^2sin2x}$ using equ. L'H $(\frac{0}{0})$

$\Rightarrow \lim_{x\to 0} \frac{-8cos2x}{24cos2x - 32sin2x - 8x^2cos2x}$

$\Rightarrow \frac{-8}{24} \Rightarrow \frac{-1}{3}$

score(solution1): Final answer 1 point. Using cosx to find sin2x 3 points. Find cos2x 1 point. Find sin2x 2 point.

score(solution2): Final answer 1 point. Using L'H 3 points. Correct 1st differential equ 1 point. Correct 2nd differential equ 2 point.

4. (14%)

(a) (7%) 導出 $\tan^{-1} x$ 在 $x = 0$ 處之泰勒展示。(需求出 第 n 項)

(b) (7%) 求 $\lim_{x\to 0} \frac{3\tan^{-1} x - 3x + x^3}{3x^5}$.

Solution:

(a) Solution 1.

$\frac{1}{1-x} = 1 + x + x^2 + ... + x^n$

$\frac{1}{1+x^2} = \frac{dtan^{-1}x}{dx} = \frac{1}{1-(-x^2)}$

$= 1 + (-x^2) + (-x^2)^2 + ... + (-x^2)^n = 1 - x^2 + x^4 - x^6 + ... + (-1)^n x^2 n$

$tan^{-1}x = \int \frac{1}{1+x^2}\, dx. = x - \frac{x^3}{3} + \frac{x^5}{5} + ... + \frac{(-1)^n}{2n+1} x^{2n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1}$

Solution 2.

$\frac{dtan^{-1}x}{dx} = (1+x^2)^{-1}$

$(1+x^2)^{-1} = \sum_{n=0}^{\infty} C_n^{-1} x^{2n}$

$tan^{-1}x = \sum_{n=0}^{\infty} C_n^{-1} \frac{x^{2n+1}}{2n+1}$

score: Final answer 1 point. Know $tan^{-1}x = \frac{1}{1-x^2}$ 2 points.

Know $\frac{1}{1-x} = 1 + x + x^2 + ...$ 2 points. Correct concept 4 points.

(b) Solution 1.

$\lim_{x\to 0} \frac{3tan^{-1}x - 3x + x^3}{3x^5} = \lim_{x\to 0} \frac{3(x - \frac{x^3}{3} + \frac{x^5}{5} + ...) - 3x + x^3}{3x^5} = \lim_{x\to 0} \frac{3(\frac{x^5}{5} - \frac{x^7}{7} + ...)}{3x^5}$

$= \frac{3}{5} * \frac{1}{3} = \frac{1}{5}.$

Solution 2.

$\lim_{x\to 0} \frac{3tan^{-1}x - 3x + x^3}{3x^5}$ because equ. satisfy L'H $(\frac{0}{0})$

$\lim_{x\to 0} \frac{3(\frac{1}{1+x^2}) - 3 + 3x^2}{15x^4} = \lim_{x\to 0} \frac{3 - 3 - 3x^2 + 3x^2 + 3x^4}{(1+x^2)15x^4} = \lim_{x\to 0} \frac{3x^4}{15x^4 + 15x^6} \Rightarrow \frac{1}{5}$

score: answer 1 point. Using equ. L'H or upward result of (a.) 6points.

5. (14%) 求出以下兩個不定積分。

(a) (7%) $\int x\sin^{-1}x dx$.

(b) (7%) $\int \frac{\ln x}{x\ln x + x}dx$.

Solution:

(a). Let $u = \sin^{-1}x, dv = xdx$ then $du = \frac{dx}{\sqrt{1-x^2}}, v = \frac{x^2}{2}$.

$$\int x\sin^{-1}x = \frac{x^2}{2}\sin^{-1}x - \frac{1}{2}\int\frac{x^2dx}{\sqrt{1-x^2}} \tag{1}$$

For integral in (1), let $x = \sin\theta$ then $dx = \cos\theta d\theta$

$$\begin{aligned}\int\frac{x^2dx}{\sqrt{1-x^2}} &= \int\frac{\sin^2\theta}{\sqrt{1-\sin^2\theta}}\cos\theta d\theta = \int\frac{\sin^2\theta}{\cos\theta}\cos\theta d\theta \\ &= \int\sin^2\theta d\theta = \int\frac{1-\cos 2\theta}{2}d\theta \\ &= \frac{\theta}{2} - \frac{\sin 2\theta}{4}\end{aligned}$$

Since $\sin\theta = x$ then $\cos\theta = \sqrt{1-x^2}$, $\sin 2\theta = \frac{2x}{\sqrt{1-x^2}}$ and $\theta = \sin^{-1}x$

$$\frac{\theta}{2} - \frac{\sin 2\theta}{4} = \frac{1}{2}\sin^{-1}x - \frac{x}{2\sqrt{1-x^2}}$$

Hence (1) become

$$\int x\sin^{-1}x = \frac{x^2}{2}\sin^{-1}x - \frac{1}{4}\sin^{-1}x + \frac{x}{4\sqrt{1-x^2}} + C$$

Grading Critiria:

- 使用分佈積分並正確令$u = sin^{-1}x, dv = xdx$得2分
- 正確寫下使用分佈積分的結果(式子(1))得1分
- 根據式子(1)的結果，令$x = \sin\theta$並正確寫下變數替換後的式子得2分
- 剩下計算過程加解答2分
- 也可以用其他方法做，給分原則不變。

(b). Let $u = \ln x$ then $du = \frac{dx}{x}$

$$\begin{aligned}\int\frac{\ln x}{x\ln x + x}dx &= \int\frac{u}{1+u}du = \int(1 - \frac{1}{1+u})du \\ &= u - \ln(1+u) + C = \ln x - \ln(1+\ln x) + C\end{aligned}$$

Grading Critiria:

- 使用變數變換，令$u = \ln x$並正確寫下變換後的式子得3分
- 剩下計算過程和解答共佔4分

6. (6%) 求出 $\sin(x^2)$ 在 $x = 0$ 處之泰勒展示的第 n 項。(可直接使用 $\sin x$ 及 $\cos x$ 之泰勒展示，不必推導。)

Solution:

Write down the tylor expansion of $\sin x$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots\cdots$$

Substitute x with x^2 then it's the tylor expansion of $sin(x^2)$

$$\sin x^2 = x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} - \frac{x^{14}}{7!} + \cdots\cdots$$

Grading Critiria:

- 寫下$\sin x$的泰勒展式得3分
- 把x換成x^2帶入$\sin x$的泰勒展式得3分
- 化簡有錯扣1分
- $\sin x$的泰勒展示寫錯直接0分

7. (7%) 求出 $\sin^2 x$ 在 $x = 0$ 處之泰勒展示的第 n 項。(可直接使用 $\sin x$ 及 $\cos x$ 之泰勒展示，不必推導。)

Solution:

Method1

$$\sin^2 x = \frac{1 - \cos 2x}{2} \qquad \text{(1 point)}$$

$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} \qquad \text{(2 points)}$$

$$\cos(2x) = \sum_{n=0}^{\infty} (-1)^n \frac{(2x)^{2n}}{(2n)!} \qquad \text{(2 points)}$$

$$\begin{aligned}\sin^2 x &= \frac{1}{2} - \frac{1}{2} \sum_{n=0}^{\infty} (-1)^n \frac{(2x)^{2n}}{(2n)!} \\ &= \frac{1}{2} \sum_{n=1}^{\infty} \frac{(-1)^{n-1} 2^{2n}}{(2n)!} x^{2n} \\ &= \sum_{n=1}^{\infty} \frac{(-1)^{n-1} 2^{2n-1}}{(2n)!} x^{2n} \qquad \text{(2 points)}\end{aligned}$$

Method2

$$\frac{d}{dx} \sin^2 x = 2 \sin x \cos x = \sin(2x) \qquad \text{(1 point)}$$

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} \qquad \text{(2 points)}$$

$$\sin(2x) = \sum_{n=0}^{\infty} (-1)^n \frac{(2x)^{2n+1}}{(2n+1)!} \qquad \text{(2 points)}$$

$$\begin{aligned}\sin^2 x &= \int_0^x \sum_{n=0}^{\infty} (-1)^n \frac{(2x)^{2n+1}}{(2n+1)!} \, dx \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n+1}}{(2n+2)!} x^{2n+2} \\ &= \sum_{n=1}^{\infty} \frac{(-1)^{n-1} 2^{2n-1}}{(2n)!} x^{2n} \qquad \text{(2 points)}\end{aligned}$$

Therefore, the n-th term of the Taylor expansion of $\sin^2 x$ centered at $x = 0$ is

$$\begin{cases} 0 & , n = 0 \\ \dfrac{(-1)^{n-1} 2^{2n-1}}{(2n)!} x^{2n} & , n \geq 1 \end{cases}$$

Note that there will be 1~2 point(s) deducted if you (to some degree) don't specify the n-th term or just leave the answer in form of $\frac{1}{2} - \frac{1}{2} \sum_{n=0}^{\infty} (-1)^n \frac{(2x)^{2n}}{(2n)!}$

8. (10%) 求曲線 $y = \ln \sec x$ 在 $0 \le x \le \frac{\pi}{4}$ 間之長度。(任何積分公式都可使用，無須推導。)

Solution:

The arc length of the curve $L = \int_a^b \sqrt{1 + [f'(x)]^2}\ dx$, (2 points)

where $f(x) = \ln \sec x$ and $f'(x) = \frac{1}{\sec x} \cdot \sec x \tan x = \tan x$ (2 points)

$$L = \int_0^{\pi/4} \sqrt{1 + \tan^2 x}\ dx \qquad \text{(2 points)}$$
$$= \int_0^{\pi/4} \sec x\ dx \qquad \text{(1 point)}$$
$$= \int_0^{\pi/4} \sec x \cdot \frac{\sec x + \tan x}{\sec x + \tan x}\ dx$$
$$= \int_0^{\pi/4} \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x}\ dx$$
$$= \ln|\sec x + \tan x|\Big|_0^{\pi/4} \qquad \text{(2 points)}$$
$$= \ln|\sqrt{2} + 1| - \ln|1 + 0|$$
$$= \ln(\sqrt{2} + 1) \qquad \text{(1 point)}$$

Note that there will be no point if you put the formula in the wrong way at the beginning.

9. (20%) 設 Ω 是由 $y = \frac{1}{x(3-x)}$, $x = 1$, $x = 2$ 及 x 軸所包圍的區域。

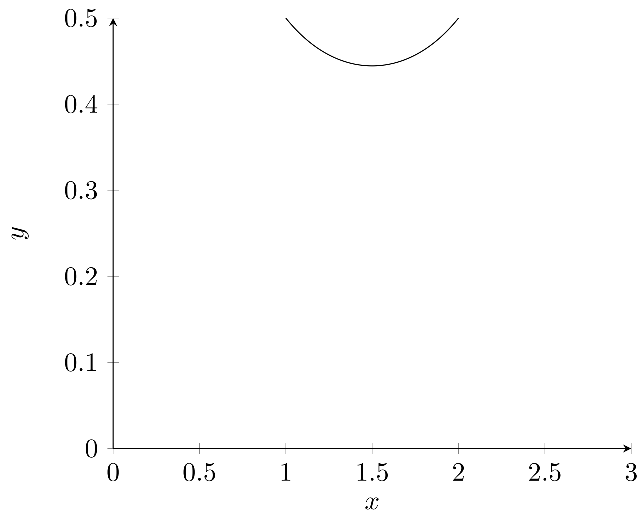
(a) (7%) 求 Ω 之面積。

(b) (7%) 求 Ω 繞 x 軸旋轉所產生的體積。

(c) (6%) 求 Ω 繞 y 軸旋轉所產生的體積。

Solution:

$$\Omega \; bounded \; by \; (y = \frac{1}{x(3-x)}) + (x = 1) + (x = 2) + (y = 0)$$



(a)

$$\int_1^2 \frac{1}{x(3-x)} dx = \int_1^2 \frac{1/3}{x} + \frac{1/3}{3-x} dx = \frac{1}{3}(ln(x))_1^2 + \frac{1}{3}(-ln(3-x))_1^2 = \frac{2}{3} ln(2)$$

Note:

$$\frac{1}{x(3-x)} = \frac{A}{x} + \frac{B}{3-x} \Rightarrow 1 = A(3-x) + Bx \Rightarrow A = 1/3 \;,\; B = 1/3$$

配分:

寫出公式 (1 pt)

求出係數 (3 pt)

積分 (2 pt)

答案 (1 pt)

(b)

法一

$$V_x = \int_1^2 \pi (\frac{1}{x(3-x)})^2 dx = \int_1^2 \pi \frac{2/27}{x} + \frac{1/9}{x^2} + \frac{2/27}{3-x} + \frac{1/9}{(3-x)^2} dx$$

$$= \pi[\frac{2}{27}(ln(x))_1^2 + \frac{1}{9}\frac{-1}{x} + \frac{2}{27}(-ln(3-x))_1^2 + \frac{1}{9}\frac{1}{3-x}] = \pi(\frac{4}{27} ln(2) + \frac{1}{9})$$

Note:

$$(\frac{1}{x(3-x)})^2 = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{3-x} + \frac{D}{(3-x)^2}$$

$$\Rightarrow 1 = Ax(3-x)^2 + B(3-x)^2 + Cx(3-x) + Dx^2$$

put $x = 0, x = 3, x = 1, x = -1, \Rightarrow A = 2/27, B = 1/9, C = 2/27, D = 1/9$

配分:
寫出公式 (1 pt)
求出係數 (3 pt)
積分 (2 pt)
答案 (1 pt)

法二
By (a) we have

$$V_x = \int_1^2 \pi(\frac{1}{x(3-x)})^2 dx = \int_1^2 \pi(\frac{1/3}{x} + \frac{1/3}{3-x})^2 dx = \frac{\pi}{9}\int_1^2 (\frac{1}{x} + \frac{1}{3-x})^2 dx = \frac{\pi}{9}\int_1^2 (\frac{1}{x^2} + \frac{2}{x(3-x)} + \frac{1}{(3-x)^2})dx$$

$$= \frac{\pi}{9}\int_1^2 \frac{1}{x^2}dx + \frac{\pi}{9}\int_1^2 \frac{2}{x(3-x)}dx + \frac{\pi}{9}\int_1^2 \frac{1}{(3-x)^2}dx = \frac{\pi}{9}(\frac{-1}{x})_1^2 + \frac{\pi}{9}\cdot 2\cdot\frac{2}{3}ln(2) + \frac{\pi}{9}(\frac{1}{(3-x)})_1^2 = \pi(\frac{4}{27}ln(2) + \frac{1}{9})$$

(c)
法一

$$V_y = \int_1^2 2\pi x f(x)dx = \int_1^2 2\pi x\frac{1}{x(3-x)}dx = \int_1^2 2\pi\frac{1}{3-x}dx = 2\pi\int_1^2 (-ln(3-x))_1^2 = 2\pi ln(2)$$

配分:
寫出公式 (1 pt)
積分 (3 pt)
答案 (2 pt)

法二
由 Pappus Thm
(對y軸旋轉之體積)=(Ω質心繞一圈軌跡長)(Ω 面積)
容易看出Ω 對x方向為對稱圖形，故質心之x座標: $\bar{x} = \frac{3}{2}$
由(a)

$$\Rightarrow V_y = 2\pi(\frac{3}{2})\cdot|\Omega| = 2\pi\cdot(\frac{3}{2})\cdot(\frac{2}{3}ln(2)) = 2\pi ln(2)$$