1. (12 points) Find the limits.

   (a) (6 points) $\lim_{n\to+\infty}\left(\frac{n}{n^2+4\cdot 1^2}+\frac{n}{n^2+4\cdot 2^2}+\frac{n}{n^2+4\cdot 3^2}+\ldots+\frac{n}{5n^2}\right)=\lim_{n\to+\infty}\sum_{i=1}^{n}\frac{n}{n^2+4i^2}$.

   (b) (6 points) $\lim_{h\to 0}\frac{1}{h}\int_{1-h}^{\sqrt[3]{1+h}}\sqrt{1+t^3}dt$.

Solution:

(a)

$$\begin{aligned}\lim_{n\to\infty}\sum_{i=1}^{n}\frac{n}{n^2+4i^2} &= \lim_{n\to\infty}\sum_{i=1}^{n}\frac{1}{1+4(\frac{i}{n})^2}\cdot\frac{1}{n}\\ &= \lim_{n\to\infty}\sum_{i=1}^{n}\frac{1}{1+4x_i^2}\cdot\Delta x\\ &= \int_0^1\frac{1}{1+4x^2}dx=\frac{1}{2}\tan^{-1}2\end{aligned}$$

where we take $a=0, b=1, \Delta x=\frac{b-a}{n}=\frac{1}{n}, x_i=a+i\Delta x=\frac{i}{n}$

Another approach:

$$\begin{aligned}\lim_{n\to\infty}\sum_{i=1}^{n}\frac{n}{n^2+4i^2} &= \lim_{n\to\infty}\sum_{i=1}^{n}\frac{1}{1+(\frac{2i}{n})^2}\cdot(\frac{1}{2}\cdot\frac{2}{n})\\ &= \lim_{n\to\infty}\sum_{i=1}^{n}\frac{1}{1+x_i^2}\cdot\frac{1}{2}\Delta x\\ &= \int_0^2\frac{1}{1+x^2}\cdot\frac{1}{2}dx=\frac{1}{2}\tan^{-1}2\end{aligned}$$

where we take $a=0, b=2, \Delta x=\frac{b-a}{n}=\frac{2}{n}, x_i=a+i\Delta x=\frac{2i}{n}$

(b) $\lim_{h\to 0}\frac{1}{h}\int_{1-h}^{\sqrt[3]{1+h}}\sqrt{1+t^3}\,dt$

$=\lim_{h\to 0}\frac{1}{1}\frac{d}{dh}\left[\int_{1-h}^{\sqrt[3]{1+h}}\sqrt{1+t^3}\,dt\right]$ $\left[\frac{0}{0}\right]$L'H' (1 points)

$=\lim_{h\to 0}\sqrt{1+(\sqrt[3]{1-h})^3}\,\frac{d}{dh}[\sqrt[3]{1+h}]-\sqrt{1+(1-h)^3}\,\frac{d}{dh}[1-h]$ (2 points)

$=\lim_{h\to 0}\sqrt{1+(\sqrt[3]{1-h})^3}\,\frac{1}{3}[1+h]^{-2/3}-\sqrt{1+(1-h)^3}\,(-1)$ (2 points)

$=\sqrt{2}\,\frac{1}{3}\times(1)+\sqrt{2}$

$=\frac{4}{3}\sqrt{2}$ (1 points)

Note:

1. If you get one derivative wrong, you will get 1 point in the total of two points.
2. If you integrate $\int\sqrt{1+t^3}dt$, you will get no point.

2. (10 points) Evaluate the integrals.

(a) (5 points) $\int \tan x \ln(\cos x)\, dx$.

(b) (5 points) $\int \frac{\sin x - 1}{\sin x \cos x}\, dx$.

Solution:

(a) (Method I)

Let $t = \cos x \Rightarrow dt = -\sin x dx$

$\Rightarrow \int \tan x \ln(\cos x) dx (2\text{pts}) = -\int \frac{\ln t}{t} dt = -\frac{1}{2}(\ln t)^2 + C (2\text{pts}) = -\frac{1}{2}(\ln(\cos x))^2 + C$ (1pt)

(Method II) Let $u = \ln\cos x, dv = \tan x dx \Rightarrow du = -\tan x dx, v = -\ln\cos x$ (2pts)

$\Rightarrow \int \tan x \ln(\cos x) dx = -(\ln\cos x)^2 - \int \tan x \ln(\cos x) dx$ (2pts)

$\Rightarrow \int \tan x \ln(\cos x) dx = -\frac{1}{2}(\ln(\cos x))^2 + C$ (1pt)

(b) (Method I)

$\int \frac{\sin x - 1}{\sin x \cos x} dx = \int \frac{1}{\cos x} - \frac{1}{\sin x \cos x} dx (2\text{pts}) = \int \sec x - \frac{2}{\sin 2x} dx$

$= \ln|\sec x + \tan x| - \int 2\csc 2x dx = \ln|\sec x + \tan x| + \ln|\csc 2x + \cot 2x| + C$ (3pts)

(Method II)

$\int \frac{\sin x - 1}{\sin x \cos x} dx = \int \frac{1}{\cos x} - \frac{1}{\sin x \cos x} dx (2\text{pts}) = \int \sec x - \frac{\sec^2 x}{\tan x} dx$

$= \ln|\sec x + \tan x| - \ln|\tan x| + C$ (3pts)

(Method III)

$\int \frac{\sin x - 1}{\sin x \cos x} dx = \int \frac{1}{\cos x} - \frac{1}{\sin x \cos x} dx$ (2pts)

$= \int \sec x - \tan x - \cot x dx = \ln|\sec x + \tan x| + \ln|\cos x| - \ln|\sin x| + C$ (3pts)

(Method IV)

$\int \frac{\sin x - 1}{\sin x \cos x} dx = \int \frac{(\sin x - 1)\cos x}{\sin x \cos^2 x} dx$

Let $t = \sin x \Rightarrow dt = \cos x dx$

$\Rightarrow \int \frac{t-1}{t(1-t^2)} dt (2\text{pts}) = \int \frac{-1}{t(t+1)} dt = \int \frac{1}{t+1} - \frac{1}{t} dt$

$= \ln\left|\frac{t+1}{t}\right| + C (2\text{pts}) = \ln\left|\frac{\sin x + 1}{\sin x}\right| + C$ (1pt)

(Method V)

Let $t = \tan\frac{x}{2} \Rightarrow dx = \frac{2}{1+t^2} dt$

$\Rightarrow \int \frac{\left(\frac{2t}{1+t^2}\right) - 1}{\left(\frac{2t}{1+t^2}\right)\left(\frac{1-t^2}{1+t^2}\right)} \frac{2}{1+t^2} dt (2\text{pts}) = \int \frac{2t - (1+t^2)}{t(1-t^2)} dt = \int \frac{(t-1)^2}{t(t+1)(t-1)} dt = \int \frac{2}{t+1} - \frac{1}{t} dt$

$= \ln\left|\frac{(t+1)^2}{t}\right| + C (2\text{pts}) = \ln\left|\frac{\left(\tan\frac{x}{2} + 1\right)^2}{\tan\frac{x}{2}}\right| + C$ (1pt)

(a) $\left(\begin{aligned} \ln|\csc 2x + \cot 2x| &= \ln|1 + \cos 2x| - \ln|\sin 2x| \\ &= \ln|2\cos^2 x| - \ln|2\sin x \cos x| = \ln|\cos x| - \ln|\sin x| \end{aligned} \right)$

(b) $\left(\ln|\sec x + \tan x| + \ln|\cos x| - \ln|\sin x| = \ln\left|\frac{1}{\sin x} + 1\right| = \ln\left|\frac{1 + \sin x}{\sin x}\right| \right)$

(c) $\left(\ln\left|\frac{\left(\tan\frac{x}{2} + 1\right)^2}{\tan\frac{x}{2}}\right| = \ln\left|\frac{\tan^2\frac{x}{2} + 2\tan\frac{x}{2} + 1}{\tan\frac{x}{2}}\right| = \ln\left|\frac{2}{\sin x} + 2\right| = \ln\left|\frac{1 + \sin x}{\sin x}\right| + \ln 2 \right)$

3. (16 points) Evaluate the integrals.

(a) (8 points) $\displaystyle\int \frac{x\,dx}{\sqrt{25-8x+x^2}}$.

(b) (8 points) $\displaystyle\int \frac{e^{2x}}{16-8e^x+e^{2x}}\,dx$.

Solution:

(a) First notice that

$$\int \frac{x\,dx}{\sqrt{25-8x+x^2}} = \underbrace{\int \frac{(x-4)\ dx}{\sqrt{9+(x-4)^2}}}_{=:I_1} + \underbrace{\int \frac{4\,dx}{\sqrt{9+(x-4)^2}}}_{=:I_2} .$$

To calculate I_1, substitute $u=(x-4)^2$ into the first integral on the right-hand-side to obtain

$$I_1 := \int \frac{(x-4)\ dx}{\sqrt{9+(x-4)^2}} = \int \frac{du}{2\sqrt{9+u}} = \sqrt{9+u}+C_1 = \sqrt{9+(x-4)^2}+C_1$$

(Method + Answer: 3+1 points)

(-0.5 points for no constant of integration or not expressing the answer in terms of x.)

To calculate I_2, let $x-4=3\tan\theta$, where $\theta\in[-\frac{\pi}{2},\frac{\pi}{2}]$, then $dx=3\sec^2\theta\,d\theta$ and $\sec\theta\geq 0$. One then also has $\sec\theta=\sqrt{1+\tan^2\theta}=\sqrt{1+\left(\frac{x-4}{3}\right)^2}=\frac{1}{3}\sqrt{25-8x+x^2}$. It follows that

$$\begin{aligned} I_2 := \int \frac{4\,dx}{\sqrt{9+(x-4)^2}} &= 4\int \frac{3\sec^2\theta\,d\theta}{3\sqrt{1+\tan^2\theta}} \\ &= 4\int |\sec\theta|\,d\theta \overset{(\text{as } \sec\theta\geq 0)}{=} 4\int \sec\theta\,d\theta \\ &= 4\ln|\sec\theta+\tan\theta|+C' \\ &= 4\ln\left|\sqrt{25-8x+x^2}+x-4\right|+C_2 . \end{aligned}$$

where $C_2 := C'-4\ln 3$. (Method + Answer: 3+1 points)

(-0.5 points for no constant of integration or not expressing the answer in terms of x.)

Therefore,

$$\int \frac{x\,dx}{\sqrt{25-8x+x^2}} = \sqrt{25-8x+x^2}+4\ln\left|\sqrt{25-8x+x^2}+x-4\right|+C ,$$

where $C := C_1+C_2$.

(No deduction even without explaining relation between different constants of integration.)

(b) Let $u=e^x$, then $du=e^x dx$. It follows that

$$\begin{aligned} \int \frac{e^{2x}\,dx}{16-8e^x+e^{2x}} &= \int \frac{u\,du}{16-8u+u^2} = \int \frac{(u-4)+4\,du}{(u-4)^2} = \int \frac{du}{u-4}+\int \frac{4\,du}{(u-4)^2} \\ &= \ln|u-4|-\frac{4}{u-4}+C = \ln|e^x-4|-\frac{4}{e^x-4}+C . \end{aligned}$$

(Method (substitution + partial fraction decomposition): 3+3 points)

(Answer (2 antiderivatives): 1+1 points)

(-0.5 points for no constant of integration or not expressing the answer in terms of x.)

4. (10 points)

(a) (5 points) Determine the values of the constant t such that $\int_1^e \frac{1}{(\ln x)^t x} dx$ is convergent. Evaluate the integral for such values of t.

(b) (5 points) Determine the values of the constant t such that $\int_e^\infty \frac{1}{(\ln x)^t x} dx$ is convergent. Evaluate the integral for such values of t.

Solution:

(a) $\int_1^e \frac{dx}{(\ln x)^t x} = \lim_{a\to 1^+} \int_a^e \frac{dx}{(\ln x)^t x}$

Let $u = \ln x$, and $du = \frac{1}{x} dx$

$$\int_a^e \frac{dx}{(\ln x)^t x} = \begin{cases} \frac{(\ln x)^{1-t}}{1-t}, & if\ t \neq 1 \\ \ln \ln x, & if\ t = 1 \end{cases}$$

$$\lim_{a\to 1^+} \int_a^e \frac{dx}{(\ln x)^t x} = \begin{cases} \lim_{a\to 1^+} (\frac{1}{1-t} - \frac{(\ln a)^{1-t}}{1-t}), & if\ t \neq 1 \\ \lim_{a\to 1^+} (-\ln \ln a), & if\ t = 1 \end{cases}$$

Therefore, if $t = 1$, the limitation is diverges, if $t > 1$ the limitation is also diverges, and if $t < 1$ the limitation is $\frac{1}{1-t}$ coverges.

Grading policies:
(1)Write down the improper integral correctly. $\Rightarrow$ 1 point.
(2)Evaluate the indefinite integral correctly. $\Rightarrow$ 2 points.
(3)Write down the conculsion and value correctly. $\Rightarrow$ 2 points.

(b) $\int_e^\infty \frac{dx}{(\ln x)^t x} = \lim_{a\to \infty} \int_e^\infty \frac{dx}{(\ln x)^t x}$

Let $u = \ln x$, and $du = \frac{1}{x} dx$

$$\int_e^\infty \frac{dx}{(\ln x)^t x} = \begin{cases} \frac{(\ln x)^{1-t}}{1-t}, & if\ t \neq 1 \\ \ln \ln x, & if\ t = 1 \end{cases}$$

$$\lim_{a\to \infty} \int_e^a \frac{dx}{(\ln x)^t x} = \begin{cases} \lim_{a\to 1^+} (\frac{(\ln a)^{1-t}}{1-t} - \frac{1}{1-t}), & if\ t \neq 1 \\ \lim_{a\to 1^+} (-\ln \ln a), & if\ t = 1 \end{cases}$$

Therefore, if $t = 1$, the limitation is diverges.
If $t > 1$, that is $1 - t < 0$. The limitation is $\frac{-1}{1-t}$ coverges.
If $t < 1$, that is $1 - t > 0$. The limitation is ∞ diverges.
So, $\int_e^\infty \frac{dx}{(\ln x)^t x}$ converges iff (if and only if) $t > 1$ and the value is $\frac{-1}{1-t}$.

Grading policies:
(1)Write down the improper integral correctly. $\Rightarrow$ 1 point.
(2)Evaluate the indefinite integral correctly. $\Rightarrow$ 2 points.
(3)Write down the conculsion and value correctly. $\Rightarrow$ 2 points.

5. (18 points) Let R be the region bounded above by the curve $y = \tan^2 x$, left by $x = 0$, below by $y = 0$, and right by $x = \pi/4$. Let $\tilde{R}$ be the region bounded above by the curve $y = \tan^p x$, left by $x = 0$, below by $y = 0$, and right by $x = \pi/2$, where $p > 0$ is a constant.

(a) (6 points) Rotate R about the x-axis. Find the resulting volume.

(b) (6 points) Rotate R about the y-axis. Find the resulting volume.

(c) (6 points) Rotate $\tilde{R}$ about the x-axis. Find the values of p such that the resulting volume is finite. (Hint: You may use the inequality $(\frac{\pi}{2} - x) \cdot \tan x < 2$, for $\frac{\pi}{4} \le x < \frac{\pi}{2}$.)

Solution:

(a) Volume $V = \int_0^{\frac{\pi}{4}} \pi y^2 dx = \pi \int_0^{\frac{\pi}{4}} \tan^4 x dx$

$$\int (\tan x)^4 dx = \int (\sec^2 x - 1) \tan^2 x dx = \int \sec^2 x \tan^2 x dx - \int \tan^2 x dx \tag{1}$$

For first integral in (1), let $u = \tan x$, $du = \sec^2 x dx$. Then

$$\int \sec^2 x \tan^2 x dx = \int u^2 du = \frac{1}{3} u^3 + C = \frac{1}{3} \tan^3 x + C \tag{2}$$

And for second integral in (1)

$$\int \tan^2 x dx = \int (\sec^2 x - 1) dx = \tan x - x + C \tag{3}$$

Substitute (2) and (3) into (1) we get

$$\int (\tan x)^4 dx = \frac{1}{3} \tan^3 x - \tan x + x + C$$

Hence the volumn V is

$$\begin{aligned} V &= \pi \Big[\frac{1}{3} \tan^3 x - \tan x + x \Big]_0^{\frac{\pi}{4}} = \pi \Big(\frac{1}{3} \tan^3 \frac{\pi}{4} - \tan \frac{\pi}{4} + \frac{\pi}{4} \Big) \\ &= \frac{\pi}{3} - \pi + \frac{\pi^2}{4} \\ &= -\frac{2\pi}{3} + \frac{\pi^2}{4} \end{aligned}$$

Grading criteria:

- Write down the formula of volumn V (either cylinder method or shell method) get 2 points.
- There are two part in the integration of $\tan^4 x$, get 3 points if you calculate both part correctly, get 1 points if you calculate only one of those correctly.
- If both part of integration are wrong, you can get 1 points if you try to simplify the integration of $\tan^4 x$
- Write down answer correctly get 1 points.

(b)

By the shell method, the resulting volume is

$$V_R = 2\pi \int_0^{\frac{\pi}{4}} x \tan^2 x dx.$$

Setting

$$u = x, \qquad dv = \tan^2 x dx = (\sec^2 x - 1)dx,$$

we have

$$du = dx, \quad v = \tan x - x.$$

Therefore,

$$V_R = 2\pi \left\{ \left[x(\tan x - x) \right]_0^{\pi/4} - \int_0^{\frac{\pi}{4}} \tan x - x dx \right\}$$

$$= 2\pi \left[x\left(\tan x - x\right) - \left(\ln|\sec x| - \frac{x^2}{2} \right) \right]_0^{\pi/4}$$

$$= \frac{\pi^2}{2} - \frac{\pi^3}{16} - \pi \ln 2$$

(1 pt: volume formula.)

(2 pts: integration by part.)

(2 pts: process.).

(1 pt: right volume.)

(c)

The volume of the given solid of revolution is given by the improper integral.

$$V = \pi \int_0^{\frac{\pi}{2}} \tan^{2p} x dx$$

$$= \pi \lim_{\theta \to (\frac{\pi}{2})^-} \int_0^{\theta} \tan^{2p} x dx$$

$$= \pi \int_0^{\frac{\pi}{4}} \tan^{2p} x dx + \pi \lim_{\theta \to (\frac{\pi}{2})^-} \int_{\frac{\pi}{4}}^{\theta} \tan^{2p} x dx.$$

Note that the volume is finite if and only if the latter integral is convergent.

(1.5 pts: correct impretation of the improper integral and convergence criterion.)

The inequality $\left(\frac{\pi}{2} - x\right) \tan x < 2$ for $\frac{\pi}{4} \le x < \frac{\pi}{2}$

implies that

$$0 < \tan^{2p} x < \left(\frac{2}{\frac{\pi}{2} - x} \right)^{2p} \quad \text{for } \frac{\pi}{4} \le x < \frac{\pi}{2}.$$

Since,

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{2^{2p}}{(\frac{\pi}{2} - x)^{2p}} dx = \lim_{\theta \to (\frac{\pi}{2})^-} \int_{\frac{\pi}{4}}^{\theta} \frac{2^{2p}}{(\frac{\pi}{2} - x)^{2p}} x dx$$

$$= \lim_{\theta \to (\frac{\pi}{2})^-} 2^{2p} \left[\frac{1}{(2p-1)(\frac{\pi}{2}-x)^{2p-1}} \right]_{\frac{\pi}{4}}^{\theta}$$

$$= \left\{ \frac{1}{(2p-1)(\frac{\pi}{2}-\theta)^{2p-1}} - \frac{1}{(2p-1)(\frac{\pi}{4})^{2p-1}} \right\}$$

$$= \begin{cases} (\frac{\pi}{4})^{1-2p} \frac{1}{1-2p} & \text{when } p < \frac{1}{2} \\ +\infty & \text{when } p > \frac{1}{2} \end{cases}.$$

It follows by the comparison test that $\int_0^{\frac{\pi}{2}} \tan^{2p} x dx$ is convergent when $p < \frac{1}{2}$.

(2 pts: showing that volume is convergent when $p < \frac{1}{2}$.)

Note also that, when $p = \frac{1}{2}$,

$$V = \pi \int_0^{\frac{\pi}{2}} \tan x dx = \lim_{\theta \to (\frac{\pi}{2})^-} \int_0^{\theta} \tan x dx = \lim_{\theta \to (\frac{\pi}{2})^-} \left[\ln |\sec x| \right]_{\frac{\pi}{4}}^{\theta} = \infty$$

(1 pt: showing that volume is divergent when $p = \frac{1}{2}$.)

when $p > \frac{1}{2}$, the volume is

$$V = \pi \int_0^{\frac{\pi}{4}} \tan^{2p} x dx + \pi \lim_{\theta \to (\frac{\pi}{2})^-} \int_{\frac{\pi}{4}}^{\theta} \tan^{2p} x dx > \pi \int_0^{\frac{\pi}{4}} \tan x dx + \pi \lim_{\theta \to (\frac{\pi}{2})^-} \int_{\frac{\pi}{4}}^{\theta} \tan \ x dx$$

Since $\tan^{2p} x > \tan x$ for $\frac{\pi}{4} \le x < \frac{\pi}{2}$,

the comparision test then assumes that the volume in question is finite if and only if $0 < p < \frac{1}{2}$.

(1.5 pts: showing that volume is divergent when $p > \frac{1}{2}$.)

6. (12 points)

(a) (6 points) Find the length of the curve

$$y = \int_0^x \sqrt{\cos(2t)}\ dt$$

from $x = 0$ to $x = \pi/4$.

(b) (6 points) Rotate the curve about the y-axis. Find the resulting surface area.

Solution:

(a)

$$\int_0^{\pi/4} \sqrt{1 + (\mathrm{d}y/\mathrm{d}x)^2}\mathrm{d}x \quad \text{2 points} \quad = \int_0^{\pi/4} \sqrt{1 + \cos(2x)}\mathrm{d}x \quad \text{1 point}$$

$$= \int_0^{\pi/4} \sqrt{2\cos^2(x)}\mathrm{d}x \quad \text{2 points}$$

$$= \sqrt{2} \int_0^{\pi/4} \cos(x)\mathrm{d}x$$

$$= \sqrt{2}\sin(x)|_0^{\frac{\pi}{4}}$$

$$= \sqrt{2}(\frac{\sqrt{2}}{2} - 0) = 1 \quad \text{1 point}$$

(b)

$$S = \int_0^{\pi/4} 2\pi x\sqrt{1 + \cos(2x)}\mathrm{d}x \quad \text{3 points} \quad = 2\pi \int_0^{\pi/4} x\sqrt{2\cos^2(x)}\mathrm{d}x$$

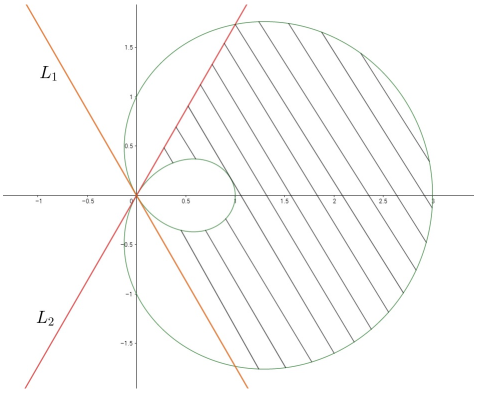
$$= 2\sqrt{2}\pi \int_0^{\pi/4} x\cos(x)\mathrm{d}x$$

$$= 2\sqrt{2}\pi\ [x\sin(x) + \cos(x)]|_0^{\frac{\pi}{4}} \quad \text{2 points}$$

$$= 2\sqrt{2}\pi[\frac{\pi}{4}(\frac{\sqrt{2}}{2}) + \frac{\sqrt{2}}{2} - 1]$$

$$= \frac{\pi^2}{2} + (2 - 2\sqrt{2})\pi \quad \text{1 point}$$

7. (10 points) The curve $C:\ r = 1 + 2\cos\theta$ and its two tangent lines, L_1 and L_2, at the pole are shown in the graph.



(a) (6 points) Find the area of the shaded region.

(b) (4 points) Now consider another curve $\tilde{C}:\ r = -1 - 2\cos(\theta - \frac{\pi}{6})$. How is the curve $\tilde{C}$ related to the curve C?

Solution:

(a) $L_1 : \theta = \frac{2\pi}{3}$ **(1%)**, $L_2 : \theta = \frac{4\pi}{3}$ **(1%)**.

$$\text{Area} = 2\left[\underset{\textbf{(1\%)}}{\int_0^{\frac{\pi}{3}}} \underset{\textbf{(1\%)}}{\frac{1}{2}(1+2\cos\theta)^2 \mathrm{d}\theta} - \underset{\textbf{(1\%)}}{\int_{\frac{2\pi}{3}}^{\pi}} \frac{1}{2}(1+2\cos\theta)^2 \mathrm{d}\theta\right]$$

$$\left(\text{or } 2\left[\int_{\frac{5\pi}{3}}^{2\pi} \frac{1}{2}(1+2\cos\theta)^2 \mathrm{d}\theta - \int_{\pi}^{\frac{4\pi}{3}} \frac{1}{2}(1+2\cos\theta)^2 \mathrm{d}\theta\right]\right)$$

$= 4\sqrt{3}$ **(1%)**

(b) Reflect through the pole **(2%)** and rotate $\frac{\pi}{6}$ (rad) counter-clockwise **(2%)**.

8. (12 points)

(a) (6 points) Solve the differential equation $x\dfrac{dy}{dx} - 2y = x^3 \tan x \sec x$, $x > 0$, and $y(\pi/3) = 0$.

(b) (6 points) Find the orthogonal trajectories of the family of curves $y = \dfrac{k}{x+1}$, where k is an arbitrary constant.

Solution:

(a)

$$\begin{aligned} &x\frac{dy}{dx} - 2y = x^3\tan(x)\sec(x) \quad x > 0, \quad y(\frac{\pi}{3}) = 0 \\ &\Rightarrow \frac{dy}{dx} - \frac{2}{x}y = x^2\tan x \sec x \end{aligned}$$

Let

$$I = \exp^{P} -\frac{2}{x} = \exp^{-2\ln|x|+c} = \frac{1}{x^2} \qquad \text{(2 points)}$$

by choosing c = 0

$$\begin{aligned} &Iy = \int \frac{1}{x^2}x^2\tan(x)\sec(x)dx = \sec(x) + c \qquad \text{(2 points)} \\ &\Rightarrow y = x^2(\sec(x) + c) \\ &\Rightarrow y(\frac{\pi}{3}) = (\frac{\pi}{3})^2(\sec(\frac{\pi}{3}) + c) = 0 \\ &\Rightarrow c = -2 \qquad \text{(1 point)} \\ &\Rightarrow y = x^2(\sec(x) - 2) \qquad \text{(1 point)} \end{aligned}$$

(b)

$$\begin{aligned} &y = \frac{k}{x+1} \Rightarrow k = y(x+1) \\ &\frac{dy}{dx} = -\frac{k}{(x+1)^2} \qquad \text{(2 points)} \\ &= -\frac{y(x+1)}{(x+1)^2} = -\frac{y}{x+1} \qquad \text{(1 point)} \end{aligned}$$

Slope field of the orthogonal trajectories

$$\begin{aligned} &\frac{dy}{dx} = \frac{x+1}{y} \qquad \text{(1 point)} \\ &\Rightarrow \int y dy = \int (x+1)dx \\ &\Rightarrow \frac{1}{2}y^2 = \frac{1}{2}x^2 + x + c \qquad \text{(2 points)} \end{aligned}$$