8.3 連續型機率

習題解答 **8.3.1.**

(1) 因為 $e^{-t^2}>0$, 且

$$\int_{-\infty}^{\infty}\frac{1}{\sqrt{\pi}}e^{-t^2}\,dt=\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty}e^{-t^2}\,dt=\frac{\sqrt{\pi}}{\sqrt{\pi}}=1$$

所以 $y=\frac{1}{\sqrt{\pi}}e^{-t^2}$ 是機率密度函數.

(2) 因為 $e^{-t}>0$, 且

$$\int_{-\infty}^{\infty}e^{-t}\,dt=\int_{0}^{\infty}e^{-t}\,dt=1$$

所以 $y=\begin{cases}e^{-t}, & t\geq 0\\ 0, & t<0\end{cases}$ 是機率密度函數.

(3) 因為 $t^{-\frac{1}{2}}e^{-\frac{t}{2}}>0$, 且

$$\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}}t^{-\frac{1}{2}}e^{-\frac{t}{2}}\,dt=\frac{1}{\sqrt{2\pi}}\int_{0^+}^{\infty}t^{-\frac{1}{2}}e^{-\frac{t}{2}}\,dt=\frac{\sqrt{2\pi}}{\sqrt{2\pi}}=1$$

所以 $y=\begin{cases}\frac{1}{\sqrt{2\pi}}t^{-\frac{1}{2}}e^{-\frac{t}{2}}, & t>0\\ 0, & t\leq 0\end{cases}$ 是機率密度函數.

(4) 因為 $t^{\alpha-1}e^{-\beta t}>0$, 且

$$\int_{-\infty}^{\infty}\frac{\beta^\alpha}{\Gamma(\alpha)}t^{\alpha-1}e^{-\beta t}\,dt=\frac{\beta^\alpha}{\Gamma(\alpha)}\int_{0}^{\infty}t^{\alpha-1}e^{-\beta t}\,dt=\frac{\beta^\alpha}{\Gamma(\alpha)}\cdot\frac{\Gamma(\alpha)}{\beta^\alpha}=1$$

所以 $y=\begin{cases}\frac{\beta^\alpha}{\Gamma(\alpha)}t^{\alpha-1}e^{-\beta t}, & t>0\\ 0, & t\leq 0\end{cases}$ 是機率密度函數.

習題解答 **8.3.2.**

因為 $e^{-\frac{(t-\mu)^2}{2\sigma^2}}>0$, 由 **習題** 8.2.6

$$\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{(t-\mu)^2}{2\sigma^2}}\,dt=\frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty}e^{-\frac{(t-\mu)^2}{2\sigma^2}}\,dt=\frac{\sqrt{2\pi}\sigma}{\sqrt{2\pi}\sigma}=1$$

所以 $\mathrm{f}_X(t)=\frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{(t-\mu)^2}{2\sigma^2}}$ 是機率密度函數.

習題解答 **8.3.3.**

由**習題** 8.2.7

$$\begin{aligned}\mathrm{E}(X)&=\int_{-\infty}^{\infty}t\cdot\frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{(t-\mu)^2}{2\sigma^2}}\,dt=\frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty}te^{-\frac{(t-\mu)^2}{2\sigma^2}}\,dt\\&=\frac{\sqrt{2\pi}\sigma\mu}{\sqrt{2\pi}\sigma}=\mu\\\mathrm{E}(X^2)&=\int_{-\infty}^{\infty}t^2\cdot\frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{(t-\mu)^2}{2\sigma^2}}\,dt=\frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty}t^2e^{-\frac{(t-\mu)^2}{2\sigma^2}}\,dt\\&=\frac{\sqrt{2\pi}\sigma(\mu^2+\sigma^2)}{\sqrt{2\pi}\sigma}=\mu^2+\sigma^2\\\mathrm{Var}(X)&=\mathrm{E}(X^2)-\mathrm{E}(X)^2=\mu^2+\sigma^2-\mu^2=\sigma^2\end{aligned}$$

習題解答 **8.3.4.**

$$\begin{aligned}\mathrm{E}(Y)&=\mathrm{E}\Big(\frac{X-\mu}{\sigma}\Big)=\frac{\mathrm{E}(X)-\mu}{\sigma}=\frac{\mu-\mu}{\sigma}=0\\\mathrm{Var}(Y)&=\mathrm{Var}\Big(\frac{X-\mu}{\sigma}\Big)=\frac{\mathrm{Var}(X)}{\sigma^2}=\frac{\sigma^2}{\sigma^2}=1\end{aligned}$$

習題解答 8.3.5.

$\lambda e^{-\lambda t}$, $t \geq 0$ 其實就是 $\Gamma(1,\lambda)$ 分配的機率密度函數. 計算期望值和變異數顯然會用到下面兩算式

$$\begin{aligned}
\textstyle\int_0^b te^{-\lambda t}\,dt &= \textstyle\int_0^b t\, d(\frac{e^{-\lambda t}}{-\lambda}) = \frac{te^{-\lambda t}}{-\lambda}\Big|_0^b + \frac{1}{\lambda}\int_0^b e^{-\lambda t}\,dt \\
\Rightarrow \quad \textstyle\int_0^\infty te^{-\lambda t}\,dt &= \lim_{b\to\infty}\Big(\frac{te^{-\lambda t}}{-\lambda}\Big|_0^b + \frac{1}{\lambda}\int_0^b e^{-\lambda t}\,dt\Big) \\
&= 0 + \tfrac{1}{\lambda}\lim_{b\to\infty} \tfrac{e^{-\lambda t}}{-\lambda}\Big|_0^b = \tfrac{1}{\lambda^2} \\
\textstyle\int_0^b t^2e^{-\lambda t}\,dt &= \textstyle\int_0^b t^2\, d(\frac{e^{-\lambda t}}{-\lambda}) = \frac{t^2e^{-\lambda t}}{-\lambda}\Big|_0^b + \frac{2}{\lambda}\int_0^b te^{-\lambda t}\,dt \\
\Rightarrow \quad \textstyle\int_0^\infty t^2e^{-\lambda t}\,dt &= \lim_{b\to\infty}\Big(\frac{t^2e^{-\lambda t}}{-\lambda}\Big|_0^b + \frac{2}{\lambda}\int_0^b te^{-\lambda t}\,dt\Big) \\
&= 0 + \tfrac{2}{\lambda}\textstyle\int_0^\infty te^{-\lambda t}\,dt = \frac{2}{\lambda}\cdot\frac{1}{\lambda^2} = \frac{2}{\lambda^3}
\end{aligned}$$

所以

$$\begin{aligned}
\mathrm{E}(X) &= \int_{-\infty}^{\infty} t\cdot\lambda e^{-\lambda t}\,dt = \lambda\int_0^\infty te^{-\lambda t}\,dt = \lambda\cdot\frac{1}{\lambda^2} = \frac{1}{\lambda} \\
\mathrm{E}(X^2) &= \int_{-\infty}^{\infty} t^2\cdot\lambda e^{-\lambda t}\,dt = \lambda\int_0^\infty t^2e^{-\lambda t}\,dt = \lambda\cdot\frac{2}{\lambda^3} = \frac{2}{\lambda^2} \\
\mathrm{Var}(X) &= \mathrm{E}(X^2) - \mathrm{E}(X)^2 = \frac{2}{\lambda^2} - (\frac{1}{\lambda})^2 = \frac{1}{\lambda^2}
\end{aligned}$$

習題解答 8.3.7.

(1) 設 $Y = X + \alpha$,

$$\begin{aligned}
F_Y(t) &\equiv \mathrm{P}(Y \leq t) = \mathrm{P}(X+\alpha \leq t) = \mathrm{P}(X \leq t-\alpha) = \textstyle\int_{-\infty}^{t-\alpha} \mathrm{f}_X(t)\,dt \\
\Rightarrow \quad \mathrm{f}_Y(t) &= F_Y'(t) = \Big(\textstyle\int_{-\infty}^{t-\alpha} \mathrm{f}_X(t)\,dt\Big)' = \mathrm{f}_X(t-\alpha)
\end{aligned}$$

(2) 設 $Y = \alpha X$,

$$\begin{aligned}
F_Y(t) &\equiv \mathrm{P}(Y \leq t) = \mathrm{P}(\alpha X \leq t) = \mathrm{P}(X \leq \tfrac{t}{\alpha}) = \textstyle\int_{-\infty}^{\frac{t}{\alpha}} \mathrm{f}_X(t)\,dt \\
\Rightarrow \quad \mathrm{f}_Y(t) &= F_Y'(t) = \Big(\textstyle\int_{-\infty}^{\frac{t}{\alpha}} \mathrm{f}_X(t)\,dt\Big)' = \mathrm{f}_X(\tfrac{t}{\alpha})\cdot(\tfrac{t}{\alpha})' = \tfrac{1}{\alpha}\mathrm{f}_X(\tfrac{t}{\alpha})
\end{aligned}$$

習題解答 8.3.9.

設 $Y = X^2$, $F_Y(t) \equiv \mathrm{P}(Y \leq t) = \mathrm{P}(X^2 \leq t)$. 當 $t \leq 0$, $F_Y(t) = 0$; 當 $t > 0$, 則

$$F_Y(t) = \mathrm{P}(-\sqrt{t} \leq X \leq \sqrt{t}) = \int_{-\sqrt{t}}^{\sqrt{t}} \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}}\,dt$$

所以當 $t \leq 0$ 時, $\mathrm{f}_Y(t) = F_Y'(t) = 0$; 當 $t > 0$,

$$\begin{aligned}
\mathrm{f}_Y(t) &= F_Y'(t) = \Big(\int_{-\sqrt{t}}^{\sqrt{t}} \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}}\,dt\Big)' \\
&= \frac{1}{\sqrt{2\pi}} e^{-\frac{t}{2}}\cdot\frac{1}{2\sqrt{t}} - \frac{1}{\sqrt{2\pi}} e^{-\frac{t}{2}}\frac{-1}{2\sqrt{t}} = \frac{1}{\sqrt{2\pi}} t^{\frac{1}{2}} e^{-\frac{t}{2}}
\end{aligned}$$

因此 $\mathrm{f}_X(t) = \begin{cases} \frac{1}{\sqrt{2\pi}} t^{-\frac{1}{2}} e^{-\frac{t}{2}}, & t > 0 \\ 0, & t \leq 0 \end{cases}$, 正是 $\chi^2(1)$ 分配.