8.2 與機率有關的瑕積分

習題解答 8.2.3.

(1) 本題假設已知 $\int_{-\infty}^{\infty} xe^{-x^2}\,dx =$

$$\lim_{b\to\infty}\int_{-b}^{b} xe^{-x^2}\,dx = \lim_{b\to\infty}\frac{e^{-x^2}}{-2}\Big|_{-b}^{b} = \lim_{b\to\infty} 0 = 0$$

(2) 本題假設已知 $\int_{-\infty}^{\infty} x^2e^{-x^2}\,dx =$

$$\begin{aligned}\int_{-b}^{b} xe^{-x^2}\,dx &= \int_{-b}^{b} x\,d(\frac{e^{-x^2}}{-2}) = \frac{xe^{-x^2}}{-2}\Big|_{-b}^{b} + \frac{1}{2}\int_{-b}^{b} e^{-x^2}\,dx \\ &= -\frac{b}{e^{b^2}} + \frac{1}{2}\int_{-b}^{b} e^{-x^2}\,dx\end{aligned}$$

所以

$$\int_{-\infty}^{\infty} x^2e^{-x^2}\,dx = \lim_{b\to\infty}(-\frac{b}{e^{b^2}} + \frac{1}{2}\int_{-b}^{b} e^{-x^2}\,dx) = 0 + \frac{\sqrt{\pi}}{2} = \frac{\sqrt{\pi}}{2}$$

習題解答 8.2.4.

(1)

$$\begin{aligned}\int_{-b}^{b} e^{\lambda x}e^{-x^2}\,dx &= \int_{-b}^{b} e^{-x^2+\lambda x}\,dx = \int_{-b}^{b} e^{-(x-\frac{\lambda}{2})^2+\frac{\lambda^2}{4}}\,dx \\ &= e^{\frac{\lambda^2}{4}}\int_{-b}^{b} e^{-(x-\frac{\lambda}{2})^2}\,dx \overset{u=x-\frac{\lambda}{2}}{=} e^{\frac{\lambda^2}{4}}\int_{-b-\frac{\lambda}{2}}^{b-\frac{\lambda}{2}} e^{-u^2}\,du\end{aligned}$$

所以

$$\begin{aligned}\int_{-\infty}^{\infty} e^{\lambda x}e^{-x^2}\,dx &= \lim_{b\to\infty}\int_{-b}^{b} e^{\lambda x}e^{-x^2}\,dx = \lim_{b\to\infty} e^{\frac{\lambda^2}{4}}\int_{-b-\frac{\lambda}{2}}^{b-\frac{\lambda}{2}} e^{-u^2}\,du \\ &= e^{\frac{\lambda^2}{4}}\cdot\int_{-\infty}^{\infty} e^{-u^2}\,du = e^{\frac{\lambda^2}{4}}\cdot\sqrt{\pi}\end{aligned}$$

(2) $\displaystyle\int_{-\infty}^{\infty} e^{\lambda x}e^{-x^2}\,dx$ 等於

$$\begin{aligned}&\int_{-\infty}^{\infty}(1+\lambda x+\frac{\lambda^2x^2}{2!}+\frac{\lambda^3x^3}{3!}+\cdots)\cdot e^{-x^2}\,dx \\ =\ &\int_{-\infty}^{\infty} e^{-x^2}\,dx + \lambda\int_{-\infty}^{\infty} xe^{-x^2}\,dx + \frac{\lambda^2}{2}\int_{-\infty}^{\infty} x^2e^{-x^2}\,dx \\ &+\frac{\lambda^3}{3!}\int_{-\infty}^{\infty} x^3e^{-x^2}\,dx + \frac{\lambda^4}{4!}\int_{-\infty}^{\infty} x^4e^{-x^2}\,dx + \cdots\end{aligned}$$

又

$$\sqrt{\pi}\cdot e^{\frac{\lambda^2}{4}} = \sqrt{\pi}\Big(1+(\frac{\lambda^2}{4})+\frac{(\frac{\lambda^2}{4})^2}{2!}+\frac{(\frac{\lambda^2}{4})^3}{3!}+\cdots\Big)$$

但 $\displaystyle\int_{-\infty}^{\infty} e^{\lambda x}e^{-x^2}\,dx = \sqrt{\pi}\cdot e^{\frac{\lambda^2}{4}}$ 比較兩邊可得

$$\begin{aligned}\int_{-\infty}^{\infty} x^{2k+1}e^{-x^2}\,dx &= 0 \\ \int_{-\infty}^{\infty} x^{2k}e^{-x^2}\,dx &= (2k)!\cdot\frac{\frac{1}{4^k}}{k!}\sqrt{\pi} = \frac{(2k)!}{4^kk!}\sqrt{\pi} = \Gamma(k+\frac{1}{2})\end{aligned}$$

習題解答 **8.2.5.**

$$\begin{aligned}\int_{-b}^{b} e^{-\lambda x^2}\,dx &= \int_{-\sqrt{\lambda}b}^{\sqrt{\lambda}b} e^{-u^2}\cdot\frac{1}{\sqrt{\lambda}}\,du \qquad (u=\sqrt{\lambda}x)\\ \Rightarrow\quad \int_{-\infty}^{\infty} e^{-\lambda x^2}\,dx &= \lim_{b\to\infty}\frac{1}{\sqrt{\lambda}}\int_{-\sqrt{\lambda}b}^{\sqrt{\lambda}b} e^{-u^2}\,du = \frac{1}{\sqrt{\lambda}}\int_{-\infty}^{\infty} e^{-u^2}\,du = \sqrt{\frac{\pi}{\lambda}}\end{aligned}$$

習題解答 **8.2.6.**

$$\begin{aligned}\int_{-b}^{b} e^{-\frac{(x-\mu)^2}{2\sigma^2}}\,dx &= \int_{-\frac{b-\mu}{\sqrt{2}\sigma}}^{\frac{b-\mu}{\sqrt{2}\sigma}} e^{-u^2}\cdot\sqrt{2}\sigma\,du \qquad (u=\frac{x-\mu}{\sqrt{2}\sigma})\\ \Rightarrow\quad \int_{-\infty}^{\infty} e^{-\frac{(x-\mu)^2}{2\sigma^2}}\,dx &= \lim_{b\to\infty}\sqrt{2}\sigma\cdot\int_{-\frac{b-\mu}{\sqrt{2}\sigma}}^{\frac{b-\mu}{\sqrt{2}\sigma}} e^{-u^2}\,du\\ &= \sqrt{2}\sigma\cdot\int_{-\infty}^{\infty} e^{-u^2}\,du = \sqrt{2\pi}\sigma\end{aligned}$$

習題解答 **8.2.7.**

(1) $\displaystyle\int_{-b}^{b} xe^{-\frac{(x-\mu)^2}{2\sigma^2}}\,dx = \int_{-\frac{b-\mu}{\sqrt{2}\sigma}}^{\frac{b-\mu}{\sqrt{2}\sigma}} (\mu+\sqrt{2}\sigma u)\cdot e^{-u^2}\cdot\sqrt{2}\sigma\,du$, 其中 $u=\dfrac{x-\mu}{\sqrt{2}\sigma}$, 則

$$\begin{aligned}\int_{-\infty}^{\infty} xe^{-\frac{(x-\mu)^2}{2\sigma^2}}\,dx &= \lim_{b\to\infty}\sqrt{2}\sigma\cdot\Big(\int_{-\frac{b-\mu}{\sqrt{2}\sigma}}^{\frac{b-\mu}{\sqrt{2}\sigma}} \mu\cdot e^{-u^2}\,du + \int_{-\frac{b-\mu}{\sqrt{2}\sigma}}^{\frac{b-\mu}{\sqrt{2}\sigma}} \sqrt{2}\sigma u e^{-u^2}\,du\Big)\\ &= \sqrt{2}\sigma\mu\lim_{b\to\infty}\int_{-\frac{b-\mu}{\sqrt{2}\sigma}}^{\frac{b-\mu}{\sqrt{2}\sigma}} e^{-u^2}\,du + 2\sigma^2\lim_{b\to\infty}\int_{-\frac{b-\mu}{\sqrt{2}\sigma}}^{\frac{b-\mu}{\sqrt{2}\sigma}} ue^{-u^2}\,du\\ &= \sqrt{2}\sigma\mu\int_{-\infty}^{\infty} e^{-u^2}\,du + 2\sigma^2\int_{-\infty}^{\infty} ue^{-u^2}\,du\\ &= \sqrt{2\pi}\sigma\mu + 0 = \sqrt{2\pi}\sigma\mu\end{aligned}$$

(2) $\displaystyle\int_{-b}^{b} x^2e^{-\frac{(x-\mu)^2}{2\sigma^2}}\,dx = \int_{-\frac{b-\mu}{\sqrt{2}\sigma}}^{\frac{b-\mu}{\sqrt{2}\sigma}} (\mu+\sqrt{2}\sigma u)^2\cdot e^{-u^2}\cdot\sqrt{2}\sigma\,du$, 其中 $u=\dfrac{x-\mu}{\sqrt{2}\sigma}$,

$$=\sqrt{2}\sigma\Big(\mu^2\int_{-\frac{b-\mu}{\sqrt{2}\sigma}}^{\frac{b-\mu}{\sqrt{2}\sigma}} e^{-u^2}\,du + 2\sqrt{2}\mu\sigma\int_{-\frac{b-\mu}{\sqrt{2}\sigma}}^{\frac{b-\mu}{\sqrt{2}\sigma}} ue^{-u^2}\,du + 2\sigma^2\int_{-\frac{b-\mu}{\sqrt{2}\sigma}}^{\frac{b-\mu}{\sqrt{2}\sigma}} u^2e^{-u^2}\,du\Big)$$

則

$$\begin{aligned}\int_{-\infty}^{\infty} x^2e^{-\frac{(x-\mu)^2}{2\sigma^2}}\,dx &= \sqrt{2}\sigma\mu^2\int_{-\infty}^{\infty} e^{-u^2}\,du + 0 + 2\sqrt{2}\sigma^3\int_{-\infty}^{\infty} u^2e^{-u^2}\,du\\ &= \sqrt{2\pi}\sigma\mu^2 + 2\sqrt{2}\sigma^3\frac{\sqrt{\pi}}{2} = \sqrt{2\pi}\sigma(\mu^2+\sigma^2)\end{aligned}$$