8.1 機率的複習與延伸

習題解答 8.1.1.

$V_2 = k$ 表示在前 $k-1$ 次裡只出現一次正面, 所以

$$\mathrm{P}(V_2 = k) = (\mathbf{C}_1^{k-1} p) \cdot p q^{k-2} = (k-1) p^2 q^{k-2}$$

又

$$\sum_{k=2}^{\infty} \mathrm{P}(V_2 = k) = \sum_{k=2}^{\infty} (k-1) p^2 q^{k-2} = p^2 \sum_{k=2}^{\infty} (k-1) q^{k-2}$$

令 $S \equiv \sum_{k=2}^{\infty} (k-1) q^{k-2} = 1 + 2q + 3q^2 + \cdots$, 則

$$\begin{aligned}
& qS = q + 2q^2 + 3q^3 \\
\Rightarrow\quad & S - qS = 1 + q + q^2 + \cdots = \frac{1}{1-q} \\
\Rightarrow\quad & S = \frac{1}{(1-q)^2} = \frac{1}{p^2}
\end{aligned}$$

因此

$$\sum_{k=2}^{\infty} \mathrm{P}(V_2 = k) = p^2 \sum_{k=2}^{\infty} (k-1) q^{k-2} = p^2 \cdot \frac{1}{p^2} = 1$$

習題解答 8.1.2.

$V_n = k$ 表示在前 $k-1$ 次裡出現 $n-1$ 次正面, 所以

$$\mathrm{P}(V_n = k) = \mathbf{C}_{n-1}^{k-1} p^{n-1} \cdot q^{k-n} \cdot p = \mathbf{C}_{n-1}^{k-1} p^n q^{k-n}$$

習題解答 8.1.3.

(1) W_1 與 W_2 獨立, 所以 $W_2 \sim W$, 且 $W_1 + W_2 = k$ 所對應的事件正好就是 $V_2 = k$ 的事件, 因此 $V = W_1 + W_2$.

$$\begin{aligned}
& \mathrm{P}(V_2 = k) = \mathrm{P}(W_1 + W_2 = k) \\
= \;& \sum_{i=1}^{k-1} \mathrm{P}(W_1 = i, W_2 = k = i) = \sum_{i=1}^{k-1} \mathrm{P}(W_1 = i) \cdot \mathrm{P}(W_2 = k - i) \\
= \;& \sum_{i=1}^{k-1} (q^{i-1} p)(q^{k-i-1} p) = \sum_{i=1}^{k-1} q^{k-2} p^2 = (k-1) q^{k-2} p^2
\end{aligned}$$

(2) $\mathrm{P}(V_n = k)$ 計算如下:

$$\begin{aligned}
& \mathrm{P}(V_n = k) = \mathrm{P}(W_1 + \cdots + W_n = k) \\
= \;& \sum_{\substack{i_1 + \cdots + i_n = k \\ i_1, \ldots, i_n \geq 1}} \mathrm{P}(W_1 = i_1, \cdots, W_n = i_n) = \sum_{\substack{i_1 + \cdots + i_n = k \\ i_1, \ldots, i_n \geq 1}} \mathrm{P}(W_1 = i_1) \cdots \mathrm{P}(W_n = i_n) \\
= \;& \sum_{\substack{i_1 + \cdots + i_n = k \\ i_1, \ldots, i_n \geq 1}} (q^{i_1 - 1} p) \cdots (q^{i_n - 1} p) = \sum_{\substack{i_1 + \cdots + i_n = k \\ i_1, \ldots, i_n \geq 1}} q^{k-n} p^n
\end{aligned}$$

但方程式 $i_1 + i_2 + \cdots + i_n = k$, 其中 $i_1,\ i_2\ \cdots i_n \geq 1$, 其整數解數目等於下式

$$x_1 + x_2 + \cdots + x_n = k - n, \quad x_1,\ x_2, \cdots x_n \geq 0$$

的整數解數目, 等於重複組合 $\mathbf{H}_{k-n}^{n} = \mathbf{C}_{k-n}^{n+k-n-1} = \mathbf{C}_{k-n}^{k-1} = \mathbf{C}_{n-1}^{k-1}$, 因此 $\mathrm{P}(V_n = k) = \mathbf{C}_{n-1}^{k-1} p^n q^{k-n}$, 和前面習題的結果相同.

$\boxed{\text{習題解答}}$ **8.1.4.**

(1) $P(X_i = 1) = \frac{2}{6} = \frac{1}{3}$.

(2) $X_i \sim B(1, \frac{1}{3}, \frac{2}{3})$, 且 X_i 彼此獨立, 所以 $Z = X_1 + \cdots + X_n$ 遵守 $B(n, \frac{1}{3}, \frac{2}{3})$ 的二項分配.

$\boxed{\text{習題解答}}$ **8.1.6.**

(1) 由定義知

$$\begin{aligned}
\mathrm{E}(W) &= 1 \cdot p + 2 \cdot qp + 3 \cdot q^2 p + 4 \cdot q^3 p + \cdots \\
q\mathrm{E}(W) &= 1 \cdot qp + 2 \cdot q^2 p + 3 \cdot q^3 p + 4 \cdot q^4 p + \cdots \\
\Rightarrow \quad (1-q)\mathrm{E}(W) &= p + qp + q^2 p + q^3 p + q^4 p \cdots \\
&= p(1 + q + q^2 + \cdots) = p \cdot \tfrac{1}{1-q} = 1 \\
\Rightarrow \quad \mathrm{E}(W) &= \tfrac{1}{1-q} = \tfrac{1}{p}
\end{aligned}$$

若發生事件 1 的機率是 $p = \frac{1}{N}$, 所以 N 次會發生一次, 符合直覺. 其他 p 類推.

(2) 由期望值性質知

$$\begin{aligned}
\mathrm{E}(V_2) &= \mathrm{E}(W_1 + W_2) = \mathrm{E}(W_1) + \mathrm{E}(W_2) = 2\mathrm{E}(W) = \frac{2}{p} \\
\mathrm{E}(V_n) &= \mathrm{E}(W_1 + \cdots + W_n) = \mathrm{E}(W_1) + \cdots + \mathrm{E}(W_n) = n\mathrm{E}(W) = \frac{n}{p}
\end{aligned}$$

$\boxed{\text{習題解答}}$ **8.1.7.**

$\mathrm{E}(Y) = \mathrm{E}(X - \mathrm{E}(X)) = \mathrm{E}(X) - \mathrm{E}(X) = 0$

$\boxed{\text{習題解答}}$ **8.1.8.**

$B(n, p, q)$ 的變異數為 npq. 但由算幾不等式, $\frac{1}{2} = \frac{p+q}{2} \geq \sqrt{pq}$, 即 $npq \leq \frac{n}{4}$, 且等號發生在 $p = q = \frac{1}{2}$ 時.

$\boxed{\text{習題解答}}$ **8.1.9.**

因為 $\mathrm{Var}(X) = pq$, 所以 $\mathrm{Var}(Z) =$

$$\mathrm{Var}(X_1 + \cdots + X_n) = \mathrm{Var}(X_1) + \cdots + \mathrm{Var}(X_n) = n\mathrm{Var}(X) = npq$$

$\boxed{\text{習題解答}}$ **8.1.10.**

$$\begin{aligned}
\mathrm{E}(Y) &= \mathrm{E}(X - \alpha) = \mathrm{E}(X) - \alpha \\
\mathrm{Var}(Y) &= \mathrm{E}((Y - \mathrm{E}(Y))^2) = \mathrm{E}(((X - \alpha) - (\mathrm{E}(X) - \alpha))^2) \\
&= \mathrm{E}((X - \mathrm{E}(X))^2) = \mathrm{Var}(X)
\end{aligned}$$

$\boxed{\text{習題解答}}$ **8.1.11.**

(1) $\mathrm{Var}(W) = \mathrm{E}(W^2) - \mathrm{E}(W)^2 = \mathrm{E}(W^2) - (\frac{1}{p})^2$

$$
\begin{array}{rcllll l}
\mathrm{E}(W^2) & = & 1^2 \cdot p+ & 2^2 \cdot qp+ & 3^2 \cdot q^2p+ & 4^2 \cdot q^3p+ & \cdots \\
q\mathrm{E}(W^2) & = & & 1^2 \cdot qp+ & 2^2 \cdot q^2p+ & 3^2 \cdot q^3p+ & \cdots \\
\Rightarrow \quad (1-q)\mathrm{E}(W^2) & = & p+ & 3qp+ & 5q^2p+ & 7q^3p+ & \cdots \\
\Rightarrow \quad \mathrm{E}(W^2) & = & 1+ & 3q+ & 5q^2+ & 7q^3+ & \cdots \\
q\mathrm{E}(W^2) & = & & q+ & 3q^2+ & 5q^3+ & \cdots \\
\Rightarrow \quad (1-q)\mathrm{E}(W^2) & = & 1 & 2q+ & 2q^2+ & 2q^3+ & \cdots
\end{array}
$$

由此得到 $(1-q)\mathrm{E}(W^2) =$

$$
-1 + 2(\frac{1}{1-q}) \quad \Rightarrow \quad \mathrm{E}(W^2) = \frac{1}{p} \cdot (-1 + \frac{2}{p}) = \frac{2}{p^2} - \frac{1}{p}
$$

所以

$$
\mathrm{Var}(W) = \mathrm{E}(W^2) - (\frac{1}{p})^2 = \frac{2}{p^2} - \frac{1}{p} - \frac{1}{p^2} = \frac{1}{p^2} - \frac{1}{p} = \frac{q}{p^2}
$$

(2) 由變異數性質知

$$
\begin{aligned}
\mathrm{Var}(V_2) &= \mathrm{Var}(W_1 + W_2) = \mathrm{Var}(W_1) + \mathrm{Var}(W_2) = 2\mathrm{Var}(W) = \frac{2q}{p^2} \\
\mathrm{Var}(V_n) &= \mathrm{Var}(\sum_{i=1}^{n} W_i) = \sum_{i=1}^{n} \mathrm{Var}(W_i) = n\mathrm{Var}(W) = \frac{nq}{p^2}
\end{aligned}
$$

$\boxed{\text{習題解答}}$ **8.1.12.**

$$
\begin{aligned}
\mathrm{E}(\bar{X}) &= \mathrm{E}\Big(\frac{X_1 + \cdots + X_n}{n}\Big) = \frac{\mathrm{E}(X_1) + \cdots + \mathrm{E}(X_n)}{n} \\
&= \frac{n\mu}{n} = \mu \\
\mathrm{Var}((\bar{}X)) &= \mathrm{Var}\Big(\frac{X_1 + \cdots + X_n}{n}\Big) = \frac{\mathrm{Var}(X_1) + \cdots + \mathrm{Var}(X_n)}{n^2} \\
&= \frac{n\sigma^2}{n} = \frac{\sigma^2}{n} \\
\mathrm{E}(S) &= \mathrm{E}\Big(\frac{\sum_{i=1}^{n}(X_i - \bar{X})^2}{n}\Big) = \frac{1}{n}\mathrm{E}\Big(\sum_{i=1}^{n} X_i^2 - 2\bar{X}\sum_{i=1}^{n} X_i + n\bar{X}^2\Big) \\
&= \frac{1}{n}\mathrm{E}\Big(\sum_{i=1}^{n} X_i^2 - 2n\bar{X} \cdot \bar{X} + n\bar{X}^2\Big) = \frac{1}{n}\mathrm{E}\Big(\sum_{i=1}^{n} X_i^2 - n\bar{X}^2\Big) \\
&= \frac{1}{n}\Big(\mathrm{E}\Big(\sum_{i=1}^{n} X_i^2\Big) - \mathrm{E}(n\bar{X}^2)\Big) = \frac{1}{n} \cdot n \cdot (\mathrm{E}(X^2) - \mathrm{E}(\bar{X}^2)\Big) \\
&= \mathrm{Var}(X) + \mathrm{E}(X)^2 - \mathrm{Var}(\bar{X}) - \mathrm{E}(\bar{X})^2 \\
&= \sigma^2 + \mu^2 - \frac{\sigma^2}{n} - \mu^2 = \frac{n-1}{n}\sigma^2
\end{aligned}
$$

$\boxed{\text{習題解答}}$ **8.1.13.**

令 $\epsilon = k\sqrt{\mathrm{Var}(X)}$. 則

$$
\begin{aligned}
\mathrm{P}(|X - \mathrm{E}(X)| \geq \epsilon) &= \mathrm{P}(|X - \mathrm{E}(X)| \geq k\sqrt{\mathrm{Var}(X)}) \\
&\leq \frac{1}{k^2} = \frac{1}{(\frac{\epsilon}{\sqrt{\mathrm{Var}(X)}})^2} = \frac{\mathrm{Var}(X)}{\epsilon^2}
\end{aligned}
$$

$\boxed{\textbf{習題解答}}$ **8.1.14.**

由題意知

$$\mathrm{P}(|\bar{X}-\frac{70}{100}|\geq\frac{1}{10})\leq\frac{\frac{pq}{n}}{(\frac{1}{10})^2}=\frac{100pq}{n}\leq\frac{100}{4n}=\frac{25}{n}$$

其中用到 $pq\leq\frac{1}{4}$. 若希望之信心為出錯的機率 $\leq\frac{1}{100}$, 則

$$\frac{25}{n}\leq\frac{1}{100}\quad\Rightarrow\quad n\geq 2500$$