7.2 一階微分方程

習題解答 7.2.3.

(1) $y' = y^2(1-y)$, 則 $y=0$ 與 $y=1$ 是解, 除此之外

$$\begin{aligned}\frac{y'}{y^2(1-y)} = 1 \quad &\Rightarrow \quad \int \frac{1}{y} + \frac{1}{y^2} + \frac{1}{(1-y)^2}\ dy = \int 1\ dt \\ &\Rightarrow \quad \ln|y| - \frac{1}{y} - \ln|1-y| = t + C \\ &\Rightarrow \quad \ln\Big|\frac{y}{1-y}\Big| - \frac{1}{y} = t + C\end{aligned}$$

$y(t)$ 以隱函數的形式出現在上式.

(3) $y' = -ty$, 則 $y=0$ 是解, 除此之外

$$\begin{aligned}\frac{y'}{y} = -t \quad &\Rightarrow \quad \int \frac{1}{y}\ dy = \int -t\ dt \ \Rightarrow\ \ln|y| = -\frac{t^2}{2} + C \\ &\Rightarrow \quad |y| = e^C e^{-\frac{t^2}{2}}\end{aligned}$$

所以 $y = C_2 e^{-\frac{t^2}{2}}$, $C_2 \in \mathbb{R}$

(5) $y' = y^2 t - y^2 = (t-1)y^2$, 則 $y=0$ 是解, 除此之外

$$\begin{aligned}\frac{y'}{y^2} = t-1 \quad &\Rightarrow \quad \int \frac{1}{y^2}\ dy = \int t-1\ dt \\ &\Rightarrow \quad -\frac{1}{y} = \frac{t^2}{2} - t + C \ \Rightarrow\ y = \frac{-1}{\frac{t^2}{2} - t + C}\end{aligned}$$

(7) $y' = t(1+y^2)$, 則

$$\begin{aligned}\frac{y'}{1+y^2} = t \quad &\Rightarrow \quad \int \frac{1}{1+y^2}\ dy = \int t\ dt \\ &\Rightarrow \quad \tan^{-1} y = \frac{t^2}{2} + C \ \Rightarrow\ y = \tan(\frac{t^2}{2} + C)\end{aligned}$$

(9) $y' = e^y \sin t$, 則

$$\begin{aligned}\frac{y'}{e^y} = \sin t \quad &\Rightarrow \quad \int e^{-y}\ dy = \int \sin t\ dt \\ &\Rightarrow \quad -e^{-y} = -\cos t + C \ \Rightarrow\ y = -\ln(\cos t - C)\end{aligned}$$

當然 t, C 要滿足 $\cos t - C > 0$ 的條件, 尤其 $C < 1$.

習題解答 7.2.4.

(2) $y' + y = \sin t$, $\int 1\ dt = t + C$, 取 $e^{\int 1\ dt}$ 為 e^t, 則

$$(e^t \cdot y)' = \sin t \cdot e^t \ \Rightarrow\ y = e^{-t}\int e^t \sin t\ dt$$

但由第三章結果知 $\displaystyle\int e^t\ dt \sin t = e^t \cdot \frac{-\cos t + \sin t}{2} + C$, 解得

$$y = e^{-t}\Big(e^t \cdot \frac{-\cos t + \sin t}{2} + C\Big) = \frac{-\cos t + \sin t}{2} + Ce^{-t}$$

(4) $y' + \ln t\, y = \ln t$, $\int \ln t\, dt = t\ln t - t + C$, 取 $e^{\int \ln t\, dt}$ 為 $e^{t\ln t - t}$, 則

$$(e^{t\ln t-t}\cdot y)' = (e^{t\ln t-t})\cdot \ln t \ \Rightarrow\ y = e^{-(t\ln t-t)}\int (e^{t\ln t-t})\cdot \ln t\, dt$$

但

$$\int (e^{t\ln t-t})\cdot \ln t\, dt = \int e^{t\ln t-t}\, d(t\ln t - t) = e^{t\ln t-t} + C$$

解得

$$y = e^{-(t\ln t-t)}(e^{e^{t\ln t-t}} + C) = 1 + Ce^{-(t\ln t-t)}$$

(6) $y' + 2ty = 4t$, $\int 2t\, dt = t^2 + C$, 取 $e^{\int 2t\, dt}$ 為 e^{t^2}, 則

$$(e^{t^2}\cdot y)' = e^{t^2}\cdot 4t \ \Rightarrow\ y = e^{-t^2}\int 4te^{t^2}\, dt = e^{-t^2}(2e^{t^2} + C) = 2 + Ce^{-t^2}$$

又 $y(0) = 5$, 則 $2 + C = 5 \Rightarrow C = 3$, 所以解得 $y = 2 + 3e^{-t^2}$

習題解答 7.2.5.

$$\begin{aligned}
V'(t) &= \sqrt{\frac{g}{\alpha}}\cdot\frac{2\sqrt{g\alpha}e^{-2\sqrt{g\alpha}t}(1+e^{-2\sqrt{g\alpha}t}) + 2\sqrt{g\alpha}e^{-2\sqrt{g\alpha}t}(1-e^{-2\sqrt{g\alpha}t})}{(1+e^{-2\sqrt{g\alpha}t})^2} \\
&= \sqrt{\frac{g}{\alpha}}\cdot\frac{4\sqrt{g\alpha}e^{-2\sqrt{g\alpha}t}}{(1+e^{-2\sqrt{g\alpha}t})^2} = \frac{4ge^{-2\sqrt{g\alpha}t}}{(1+e^{-2\sqrt{g\alpha}t})^2} \\
g - \alpha v^2 &= g - \alpha\Big(\sqrt{\frac{g}{\alpha}}\ \frac{1-e^{-2\sqrt{g\alpha}\,t}}{1+e^{-2\sqrt{g\alpha}\,t}}\Big)^2 = g\cdot\Big(1 - \Big(\frac{1-e^{-2\sqrt{g\alpha}\,t}}{1+e^{-2\sqrt{g\alpha}\,t}}\Big)^2\Big) \\
&= g\cdot\frac{2\cdot 2\cdot e^{-2\sqrt{g\alpha}t}}{(1+e^{-2\sqrt{g\alpha}t})^2} = \frac{4ge^{-2\sqrt{g\alpha}t}}{(1+e^{-2\sqrt{g\alpha}t})^2}
\end{aligned}$$

兩式相同.